

# Full wave analysis of ICRF waves and Alfvén eigenmodes in toroidal plasmas

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## Abstract

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In order to investigate the behavior of ICRF waves and Alfvén eigenmodes in toroidal plasmas, such as tokamaks and toroidal helical devices, we have revised the full wave code TASK/WM to deal with the plasma with three-dimensional inhomogeneity. We solved Maxwell's equation as a boundary-value problem in the flux coordinates and the response of the plasma is described by a dielectric tensor including kinetic effects. First we analyze propagation and absorption of the ICRF waves in LHD plasmas. Next we analyze the Alfvén eigenmodes both within and below the toroidicity-induced frequency gap in tokamaks. The effects of toroidal plasma rotation and reversed magnetic shear are studied. Mode-structure of RTAE and excitation by energetic ions are also studied.

# Linear Stability Analysis of Alfvén Eigenmode

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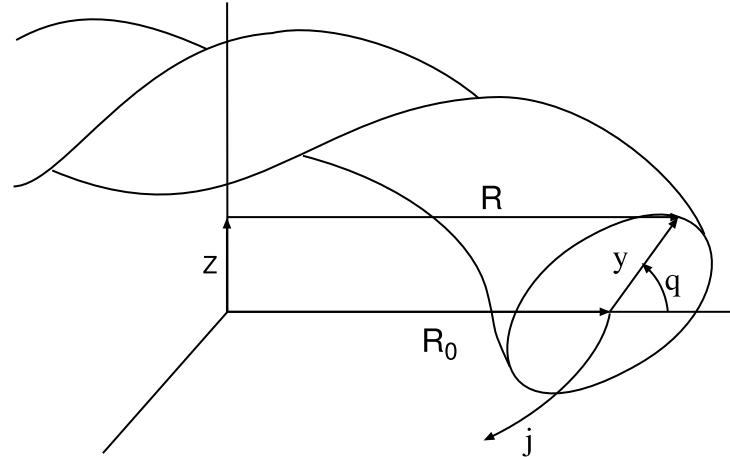
- **MHD Analysis** (Ideal, Resistive)
- **MHD including Kinetic Effect** (perturbative)
  - Eigen function from MHD analysis, Growth rate including kinetic effects
- **Kinetic Analysis** (Electron thermal motion, Ion gyromotion, Drift motion)
  - PENN code (Jaun, Alfvén Lab)
  - TASK/WM (Fukuyama)
- **Ballooning Expansion** (High  $n$  mode)
  - HINST (Gorelenkov, Cheng)
  - 2D-WKB (Vlad, Chen, Zonka)
- **3D Full Wave Code: TASK/WM**
  - Magnetic surface coordinates from MHD Equilibrium Analysis
  - Boundary value problem of Maxwell's equation. Dielectric tensor)
  - Fourier mode expansion in poloidal and toroidal direction, FDM in radius)
  - Looking for complex eigen frequency which maximize the integral of wave field.

# Full Wave Code: TASK/WM

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- **Magnetic Flux Coordinates (Non-Orthogonal)**

- Minor radius direction: Poloidal Magnetic Flux  $\psi$
- Poloidal direction:  $\theta$
- Toroidal direction:  $\varphi$



- **Co-variant expression of  $E$**

$$\mathbf{E} = E_1 \mathbf{e}^1 + E_2 \mathbf{e}^2 + E_3 \mathbf{e}^3$$

where contra-variant basis

$$\mathbf{e}^1 = \nabla\psi, \quad \mathbf{e}^2 = \nabla\theta, \quad \mathbf{e}^3 = \nabla\varphi$$

- **$J$  : Jacobian** 
$$J = \frac{1}{\mathbf{e}^1 \cdot \mathbf{e}^2 \times \mathbf{e}^3} = \frac{1}{\nabla\psi \cdot \nabla\theta \times \nabla\varphi}$$

- **$g$  : Metric tensor** 
$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$$
, where co-variant basis  $\mathbf{e}_i \equiv \partial\mathbf{r}/\partial x_i$

# Wave Equation

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- **Maxwell's equation** for stationary wave electric field  $\mathbf{E}$   
(angular frequency  $\omega$ , light velocity  $c$ )

$$\nabla \times \nabla \times \mathbf{E} = \frac{\omega^2 \leftrightarrow}{c^2} \epsilon \cdot \mathbf{E} + i\omega\mu_0 \mathbf{j}_{\text{ext}}$$

- $\overset{\leftrightarrow}{\epsilon}$  : **Dielectric Tensor** : Effects of finite temperature  
**Cyclotron damping, Landau damping**

- $\mathbf{j}_{\text{ext}}$  : Antenna Current

- **Wave Equation in Non-Orthogonal Coordinates** (radial components)

$$(\nabla \times \nabla \times \mathbf{E})^1 = \frac{1}{J} \left[ \frac{\partial}{\partial x^2} \left\{ \frac{g_{31}}{J} \left( \frac{\partial E_3}{\partial x^2} - \frac{\partial E_2}{\partial x^3} \right) + \frac{g_{32}}{J} \left( \frac{\partial E_1}{\partial x^3} - \frac{\partial E_3}{\partial x^1} \right) + \frac{g_{33}}{J} \left( \frac{\partial E_2}{\partial x^1} - \frac{\partial E_1}{\partial x^2} \right) \right\} \right. \\ \left. - \frac{\partial}{\partial x^3} \left\{ \frac{g_{21}}{J} \left( \frac{\partial E_3}{\partial x^2} - \frac{\partial E_2}{\partial x^3} \right) + \frac{g_{22}}{J} \left( \frac{\partial E_1}{\partial x^3} - \frac{\partial E_3}{\partial x^1} \right) + \frac{g_{23}}{J} \left( \frac{\partial E_2}{\partial x^1} - \frac{\partial E_1}{\partial x^2} \right) \right\} \right]$$

- $(x^1, x^2, x^3) = (\psi, \theta, \varphi)$
- Similar expression for poloidal and toroidal components

# Response of Plasmas

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- Usually the dielectric tensor  $\overset{\leftrightarrow}{\epsilon}$  is calculated in Cartesian coordinates with static magnetic field along the  $z$  axis.
- **Local normalized orthogonal coordinates**

$$\hat{\mathbf{e}}_s = \frac{\nabla \psi}{|\nabla \psi|}, \quad \hat{\mathbf{e}}_b = \hat{\mathbf{e}}_h \times \hat{\mathbf{e}}_\psi, \quad \hat{\mathbf{e}}_h = \frac{\mathbf{B}_0}{|\mathbf{B}_0|}$$

- Variable Transformation:  $\overset{\leftrightarrow}{\mu}$

$$\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \overset{\leftrightarrow}{\mu} \cdot \begin{pmatrix} E_s \\ E_b \\ E_h \end{pmatrix}$$

$$\overset{\leftrightarrow}{\mu} \equiv \begin{pmatrix} \frac{1}{\sqrt{g^{11}}} & \frac{d}{\sqrt{Jg^{11}}} & c_2 g_{12} + c_3 g_{13} \\ 0 & c_3 J \sqrt{g^{11}} & c_2 g_{22} + c_3 g_{23} \\ 0 & -c_2 J \sqrt{g^{11}} & c_2 g_{32} + c_3 g_{33} \end{pmatrix} \quad \begin{aligned} c_2 &= B^\theta / B, & c_3 &= B^\phi / B \\ d &= c_2(g_{23}g_{12} - g_{22}g_{31}) + c_3(g_{33}g_{12} - g_{32}g_{31}) \\ g^{11} &= (g_{22}g_{33} - g_{23}g_{32}) / J^2 \end{aligned}$$

- **Dielectric Tensor in Non-Orthogonal Coordinates:**

$$\overset{\leftrightarrow}{\epsilon} = \overset{\leftrightarrow}{\mu} \cdot \overset{\leftrightarrow}{\epsilon}_{sbh} \cdot \overset{\leftrightarrow}{\mu}^{-1}$$

# Fourier Mode Expansion

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- Fourier Expansion in Poloidal and Toroidal Directions
- Spatial variation of wave electric field, medium and the L.H.S. of Maxwell's equation

$$E(\psi, \theta, \varphi) = \sum_{mn} E_{mn}(\psi) e^{i(m\theta+n\varphi)}$$

$$G(\psi, \theta, \varphi) = \sum_{lk} G_{lk}(\psi) e^{i(l\theta+kN_p\varphi)}$$

$$J(\nabla \times \nabla \times \mathbf{E}) = G(\psi, \theta, \varphi) E(\psi, \theta, \varphi) = \sum_{m'n'} [J(\nabla \times \nabla \times \mathbf{E})]_{m'n'} e^{i(m'\theta+n'\varphi)}$$

- Coupling between various modes ( $N_h$  : Rotation number of helical coil in  $\varphi$ )

Mode Number	Toroidal Direction	Poloidal Direction
Wave electric field $\mathbf{E}$	$n$	$m$
Medium $G$	$kN_h$	$l$
$J(\nabla \times \nabla \times \mathbf{E})$	$n'$	$m'$
Relations	$n' = n + kN_h$	$m' = m + l$

# Parallel Wave Number

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- **Dielectric tensor**  $\overset{\leftrightarrow}{\epsilon}(\psi, \theta, \varphi, k_{\parallel}^{m''n''})$  depends on **parallel wave number**  $k_{\parallel}^{m'',n''}$  through the **plasma dispersion function**  $Z[(\omega - N\omega_{\text{cs}})/k_{\parallel}^{m''n''}v_{\text{Ts}}]$

$$\begin{aligned} k_{\parallel}^{m'',n''} &= -i\hat{\mathbf{e}}_h \cdot \nabla = -i\hat{\mathbf{e}}_h \cdot (\nabla\theta \frac{\partial}{\partial\theta} + \nabla\varphi \frac{\partial}{\partial\varphi}) \\ &= -i\hat{\mathbf{e}}_h \cdot (\mathbf{e}^2 \frac{\partial}{\partial\theta} + \mathbf{e}^3 \frac{\partial}{\partial\varphi}) = m'' \frac{B^\theta}{|B|} + n'' \frac{B^\varphi}{|B|} \end{aligned}$$

- **Fourier components of electric displacement**

$$(J \overset{\leftrightarrow}{\epsilon} \cdot \mathbf{E})^i = J \overset{\leftrightarrow}{g}^{-1} \cdot \overset{\leftrightarrow}{\mu} \cdot \overset{\leftrightarrow}{\epsilon}_{sbh} \cdot \overset{\leftrightarrow}{\mu}^{-1} \cdot \mathbf{E}_i$$

$m'$	$\ell_3$	$\ell_2$	$\ell_1$	$m$
$n'$	$k_3$	$k_2$	$k_1$	$n$

therefore

$$\begin{aligned} m'' &= m + \ell_1 + \frac{1}{2}\ell_2 & n'' &= n + k_1 + \frac{1}{2}k_2 \\ m' &= m + \ell_1 + \ell_2 + \ell_3 & n' &= n + k_1 + k_2 + k_3 \end{aligned}$$

# Destabilization by Energetic Ion

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- Drift kinetic equation

$$\left[ \frac{\partial}{\partial t} + v_{\parallel} \nabla_{\parallel} + (\mathbf{v}_d + \mathbf{v}_E) \cdot \nabla + \frac{e_{\alpha}}{m_{\alpha}} (v_{\parallel} E_{\parallel} + \mathbf{v}_d \cdot \mathbf{E}) \frac{\partial}{\partial \varepsilon} \right] f_{\alpha} = 0$$

where

$$\varepsilon = \frac{1}{2} m_{\alpha} v^2, \quad \mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2}, \quad \mathbf{v}_d = v_d \sin \theta \hat{\mathbf{r}} + v_d \cos \theta \hat{\boldsymbol{\theta}}, \quad v_d = \frac{m_{\alpha}}{e_{\alpha} B R} \cdot \frac{v_{\perp}^2}{2 + v_{\parallel}^2}$$

- Anti-Hermite part of electric susceptibility tensor

$$\begin{aligned} \overset{\leftrightarrow}{\chi}_{mm'} &= \begin{pmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} P_{m-1,m-2} \delta_{m',m-2} \\ &+ \begin{pmatrix} 0 & 0 & Q_{m-1,m-1} \\ 0 & 0 & -i Q_{m-1,m-1} \\ Q_{m,m-1} & -i Q_{m,m-1} & 0 \end{pmatrix} \delta_{m',m-1} \\ &+ \begin{pmatrix} (P_{m-1,m} + P_{m+1,m}) & i(P_{m-1,m} - P_{m+1,m}) & 0 \\ -i(P_{m-1,m} - P_{m+1,m}) & (P_{m-1,m} + P_{m+1,m}) & 0 \\ 0 & 0 & R_{m-1,m-1} \end{pmatrix} \delta_{m',m} \end{aligned}$$

$$\begin{aligned}
& + \begin{pmatrix} 0 & 0 & Q_{m+1,m+1} \\ 0 & 0 & iQ_{m+1,m+1} \\ Q_{m,m+1} & iQ_{m,m+1} & 0 \end{pmatrix} \delta_{m',m+1} \\
& + \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} P_{m+1,m+2} \delta_{m',m+2}
\end{aligned}$$

- In the case of Maxwellian velocity distribution

$$P_{m,m'} = i \frac{\omega_{p\alpha}^2}{2\omega^2} \left(1 - \frac{\omega_{*\alpha m'}}{\omega}\right) \frac{\rho_\alpha^2}{R^2} \sqrt{\pi} x_m \left(\frac{1}{2} + x_m^2 + x_m^4\right) e^{-x_m^2}$$

$$Q_{m,m'} = i \frac{\omega_{p\alpha}^2}{2\omega^2} \left(1 - \frac{\omega_{*\alpha m'}}{\omega}\right) \frac{\rho_\alpha}{R} \sqrt{\pi} 2x_m^2 \left(\frac{1}{2} + x_m^2\right) e^{-x_m^2}$$

$$R_{m,m'} = i \frac{\omega_{p\alpha}^2}{2\omega^2} \left(1 - \frac{\omega_{*\alpha m'}}{\omega}\right) \sqrt{\pi} 4x_m^3 e^{-x_m^2}$$

$$x_m = \omega / |k_{\parallel m}| v_{T\alpha},$$

$$\rho_\alpha = v_{T\alpha} / \omega_{c\alpha},$$

$$v_{T\alpha} = \sqrt{2T_\alpha / m_\alpha}$$

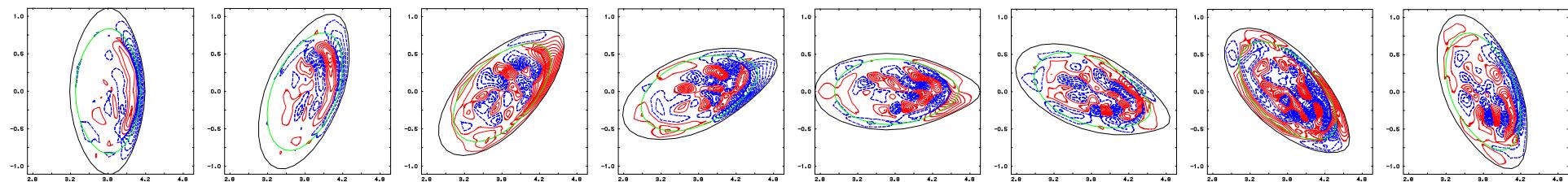
# ICRF Waves in Toroidal Helical Plasmas (Cold Plasma Model)

**LHD** ( $B_0 = 3\text{ T}$ ,  $R_0 = 3.8\text{ m}$ )

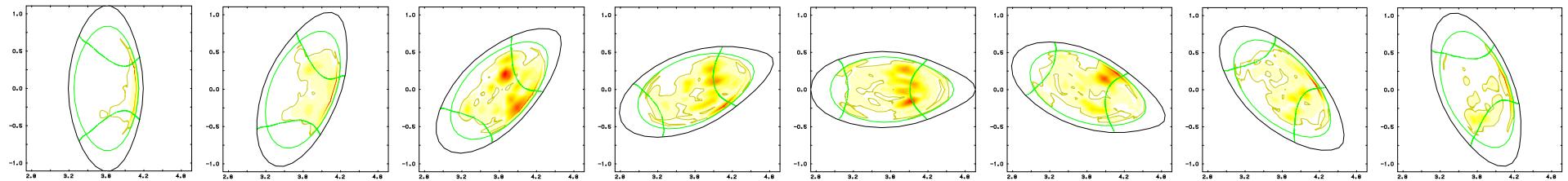
$f = 42\text{ MHz}$ ,  $n_{\phi 0} = 20$ ,  $n_{e0} = 3 \times 10^{19}\text{ m}^{-3}$ ,  $n_{\text{H}}/(n_{\text{He}} + n_{\text{H}}) = 0.235$ ,

$N_{r\max} = 100$ ,  $N_{\theta\max} = 16$  ( $m = -7 \dots 7$ ),  $N_{\phi\max} = 4$  ( $n = 10, 20, 30$ )

**Wave electric field** (imaginary part of poloidal component)



**Power deposition profile** (minority ion)



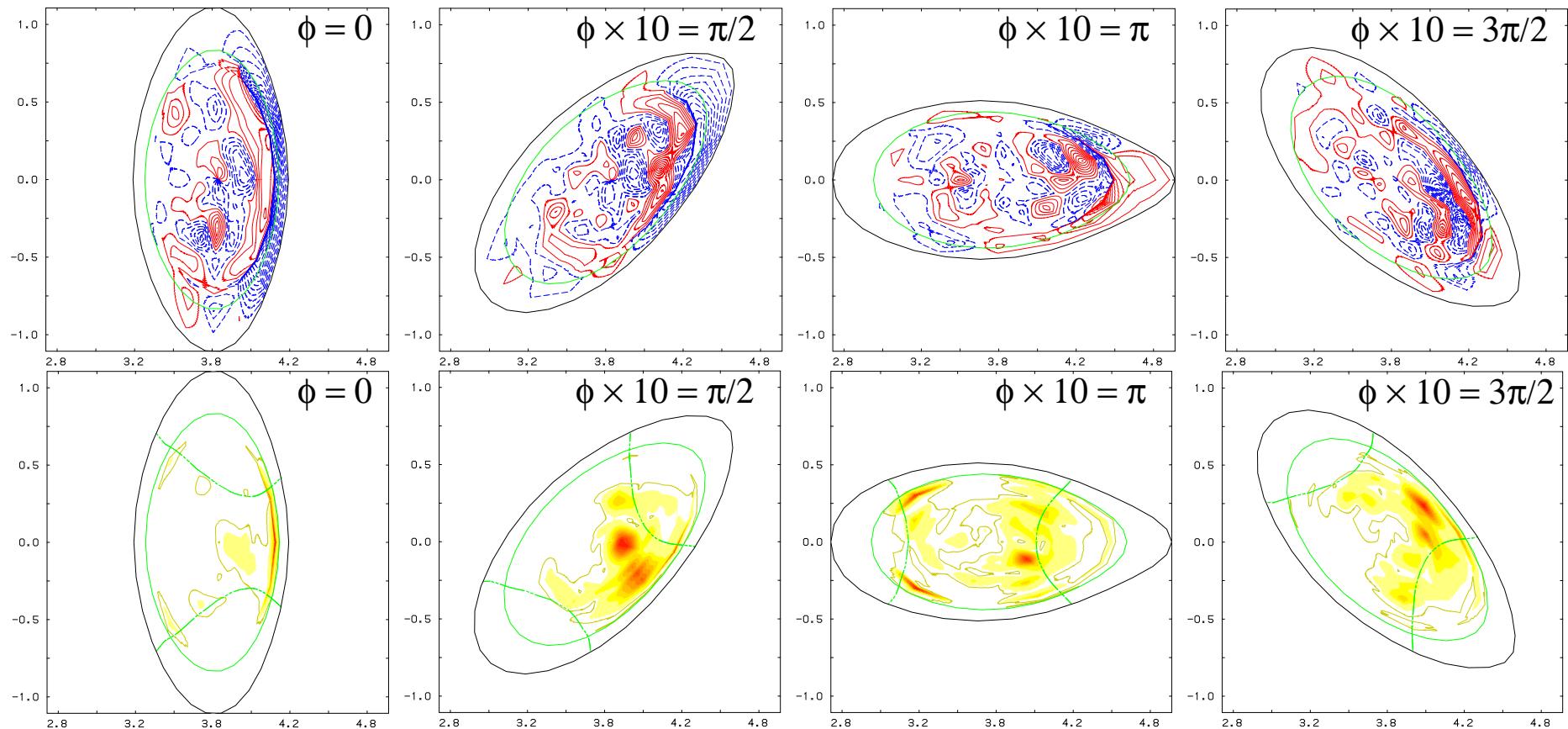
# Typical Poloidal Profile

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**LHD** ( $B_0 = 3\text{ T}$ ,  $R_0 = 3.8\text{ m}$ )

$f = 42\text{ MHz}$ ,  $n_{\phi 0} = 20$ ,  $n_{e0} = 3 \times 10^{19}\text{ m}^{-3}$ ,  $n_{\text{H}}/(n_{\text{He}} + n_{\text{H}}) = 0.235$ ,

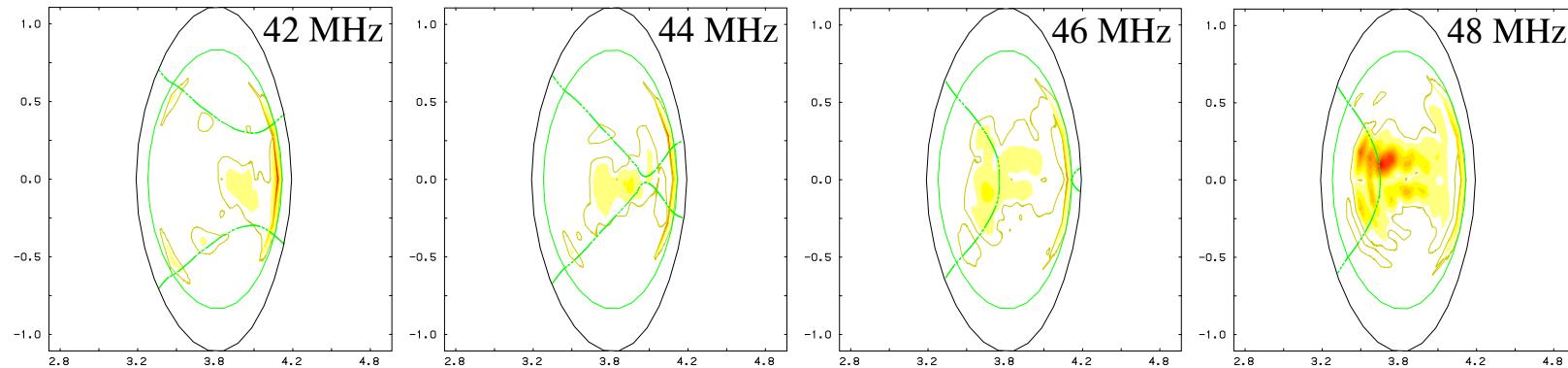
$N_{r\max} = 100$ ,  $N_{\theta\max} = 16$  ( $m = -7 \dots 7$ ),  $N_{\phi\max} = 4$  ( $n = 10, 20, 30$ )



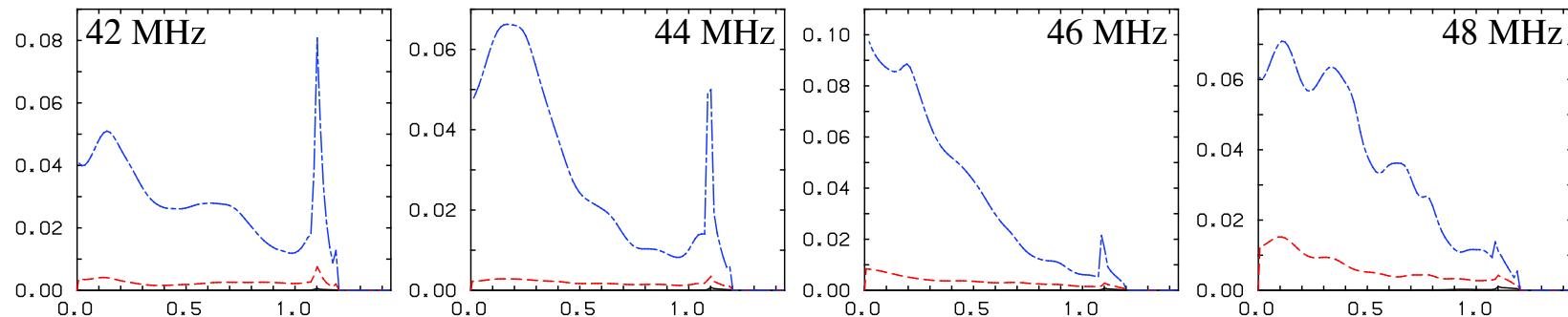
# Frequency Dependence

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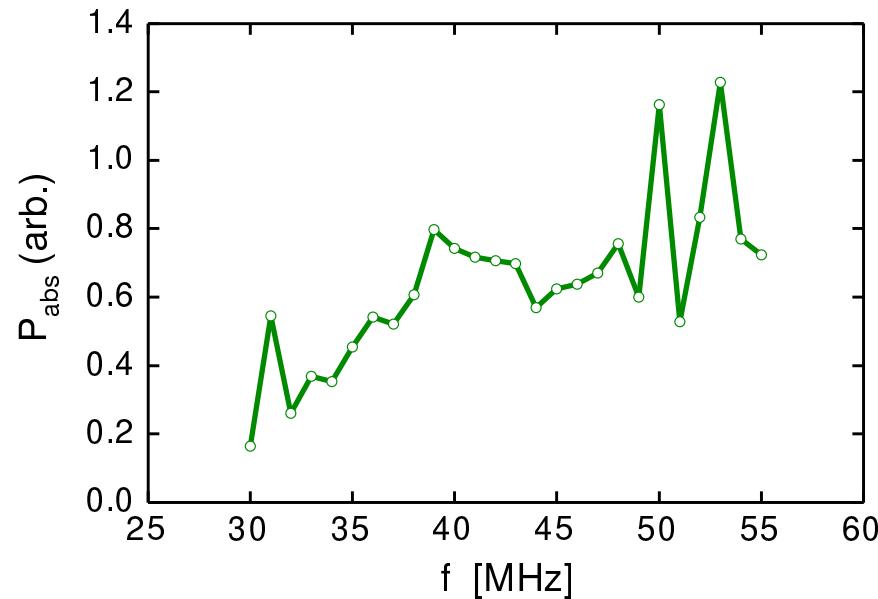
- Power deposition profile



- Radial deposition profile



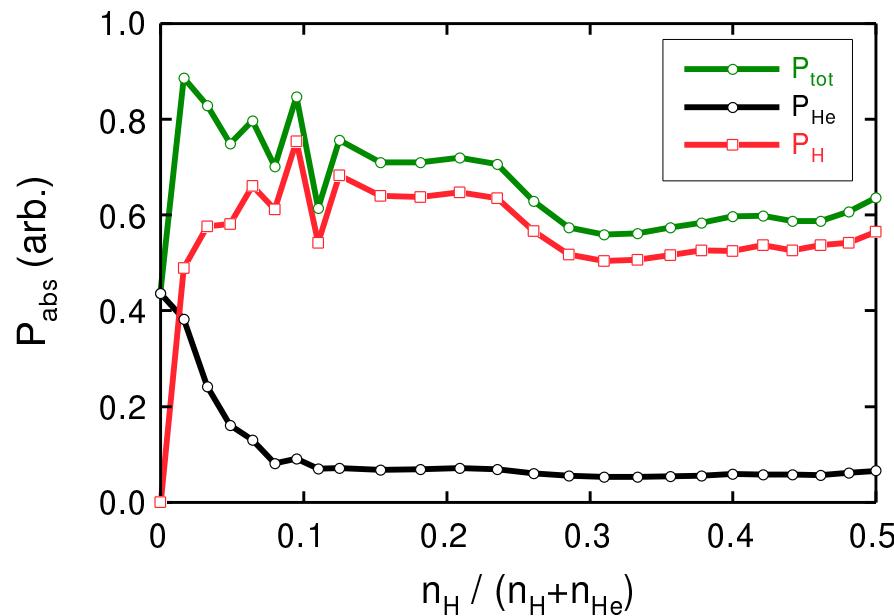
- Total power absorption



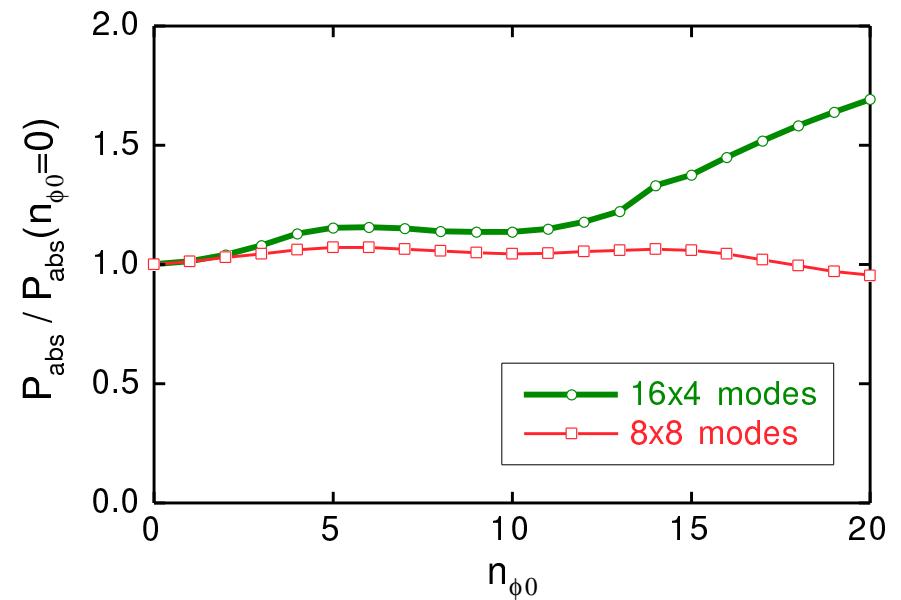
# Parameter Dependence

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Dependence on  $n_{\text{H}}$  Ratio



Dependence on  $n_{\phi 0}$

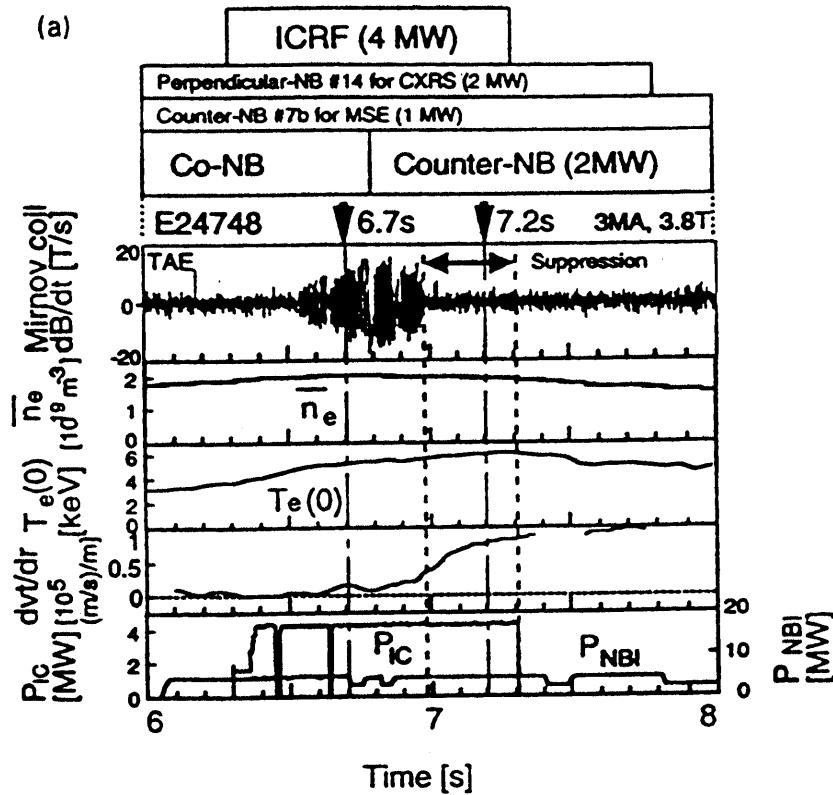


# Effect of Toroidal Plasma Rotation

## Experimental Results on JT-60U

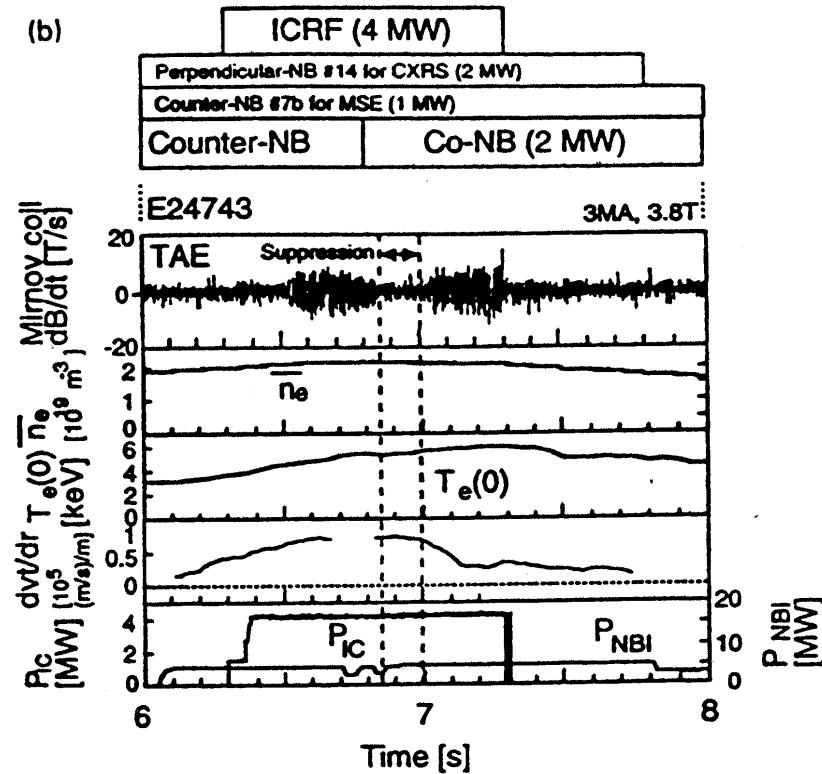
Ref. M. Saigusa et al., Nucl. Fusion, 37 (1997) 1559.

Co-NBI  $\longrightarrow$  Counter-NBI



Counter-NBI: stabilization

Counter-NBI  $\longrightarrow$  Co-NBI



Co-NBI: destabilization

# Dispersion Relation including Toroidal Rotation

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- Dispersion relation

$$\left(k_{||m}^2 - \frac{(\omega - k_{||m}u)^2}{v_A^2}\right) \left(k_{||m+1}^2 - \frac{(\omega - k_{||m+1}u)^2}{v_A^2}\right) - \epsilon^2 \frac{(\omega - k_{||m}u)^2(\omega - k_{||m+1}u)^2}{v_A^4} = 0$$

- Parallel wave number  $k_{||m} = \frac{1}{R} \left(n + \frac{m}{q}\right)$
- Alfvén resonance condition without toroidal effect

$$\omega^2 = k_{||m}^2(u \pm v_A)^2, \quad \omega^2 = k_{||m+1}^2(u \pm v_A)^2$$

- Condition for frequency gap

$$k_{||m}(u - v_A) = k_{||m+1}(u + v_A)$$

- Safety factor at TAE gap:  $q$

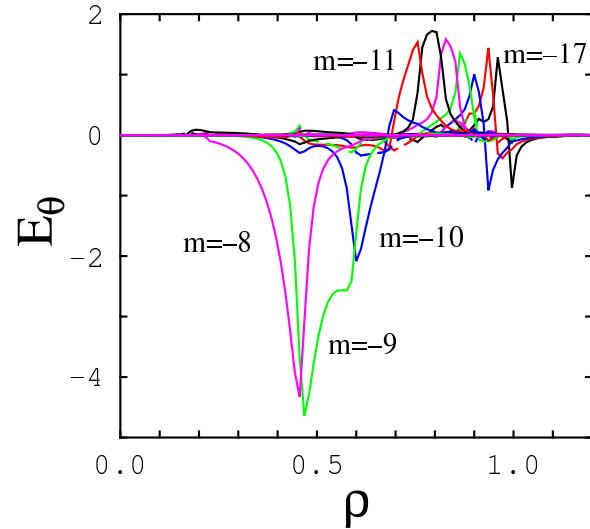
$$q = -\frac{m + 1/2}{n} - \frac{1}{2n} \frac{u}{v_A}$$

- TAE gap frequency  $\omega$ : parabolic with respect to  $u$

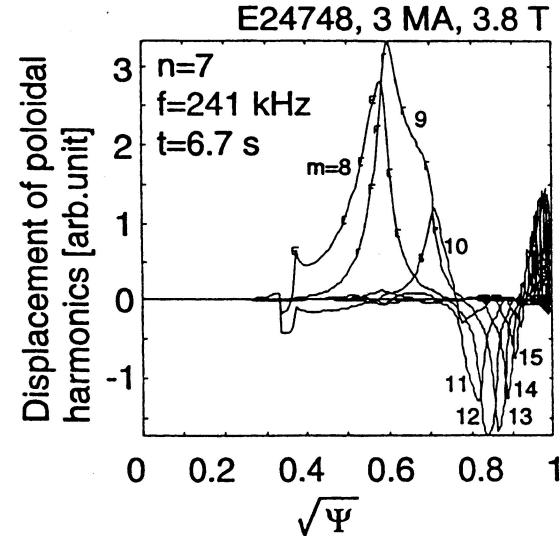
$$\omega = \frac{v_A}{2qR} \left(1 - \frac{u^2}{v_A^2}\right)$$

# Effect of Rotation on $n = 7$ mode

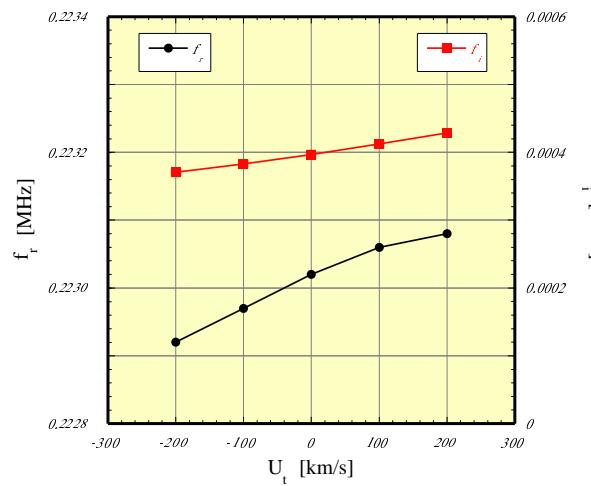
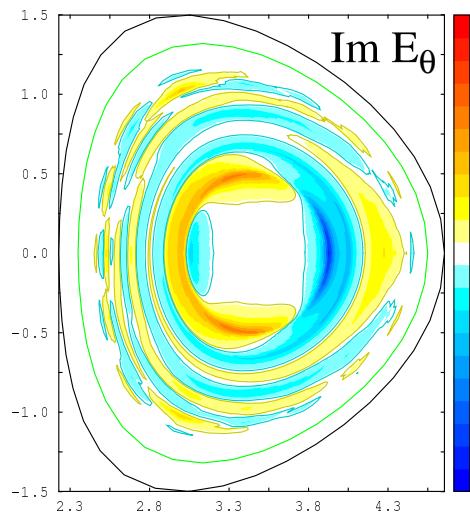
- $n = 7, m = -17 \sim -3, f = 223 \text{ kHz}$



Shape of eigen function agree with Nova-K



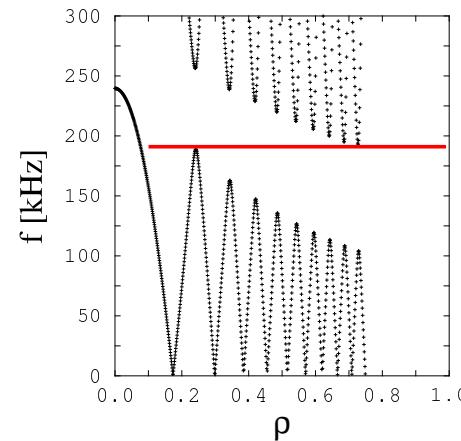
- Rotation velocity dependence: Stabilizing for co rotation (Contradict with exp.)



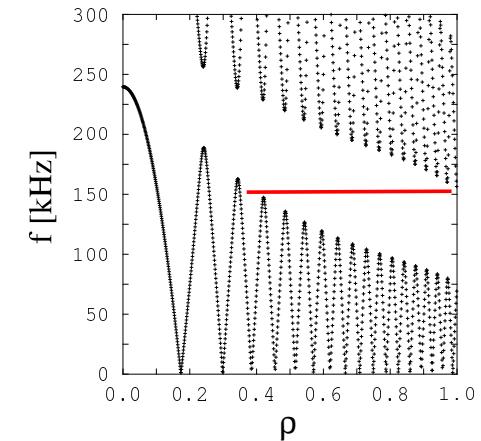
# Influence of poloidal mode range : $n = 7$ mode

- Radial structure of Alfvén Continuum

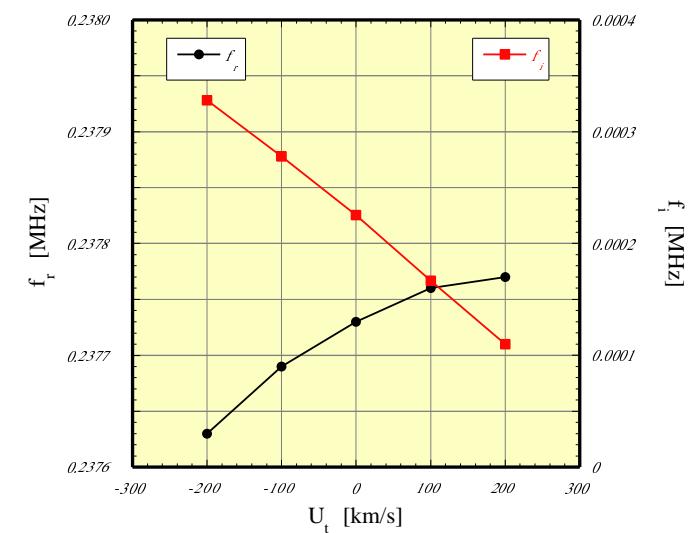
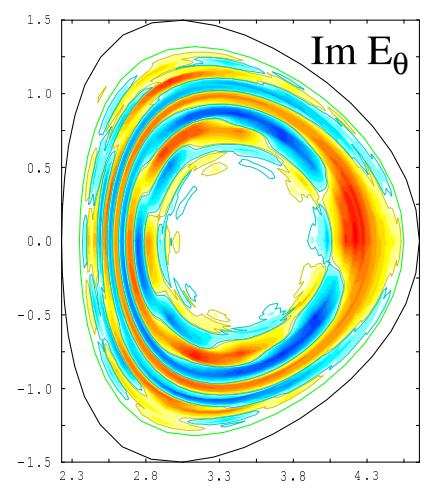
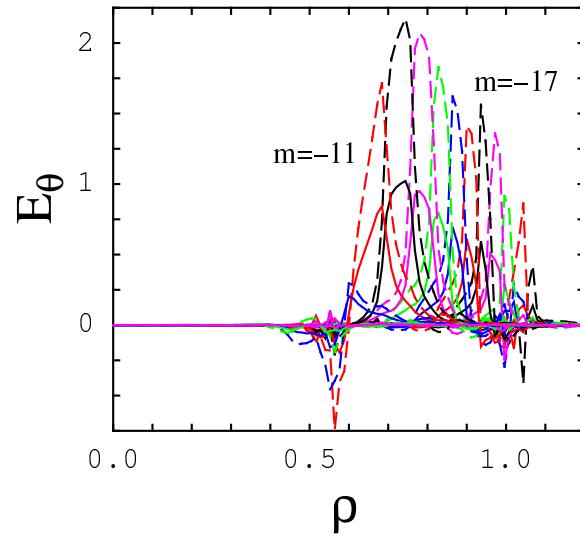
$m = -17 \sim -3$



$m = -21 \sim -7$



- $n = 7, m = -21 \sim -7, f = 238$  kHz : Destabilizing for co-rotation (agree with exp.)

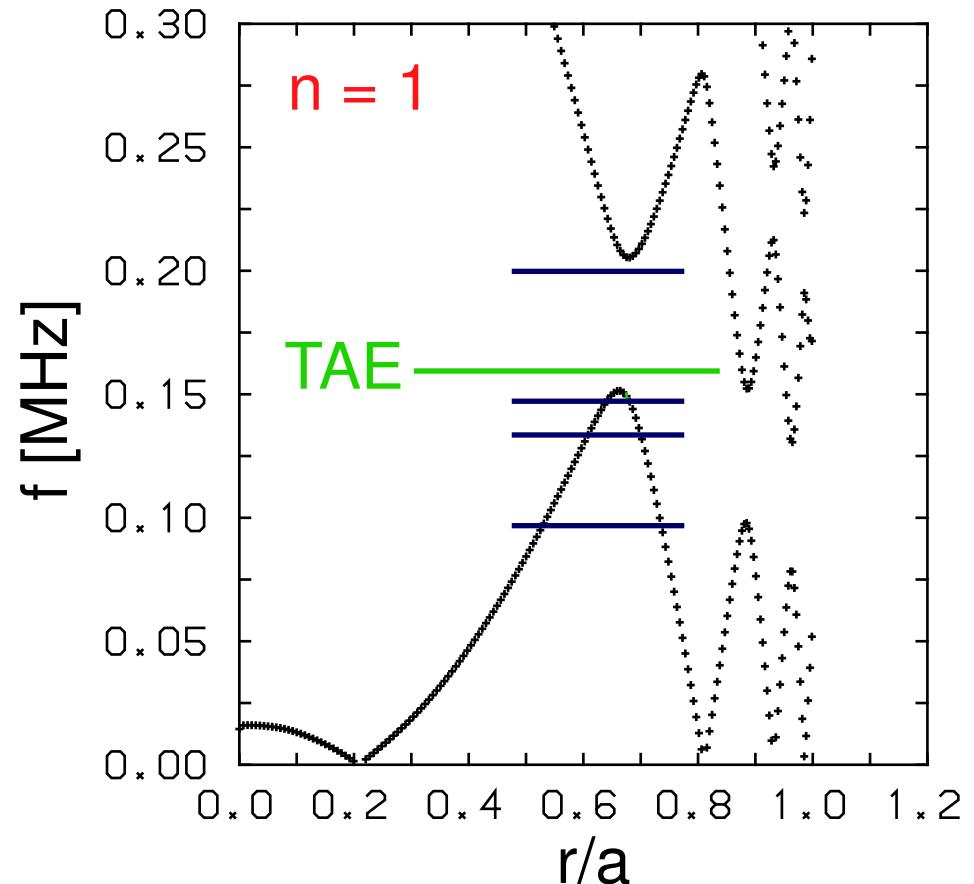


# Alfvén Eigenmodes Below Gap frequency

## Parameters

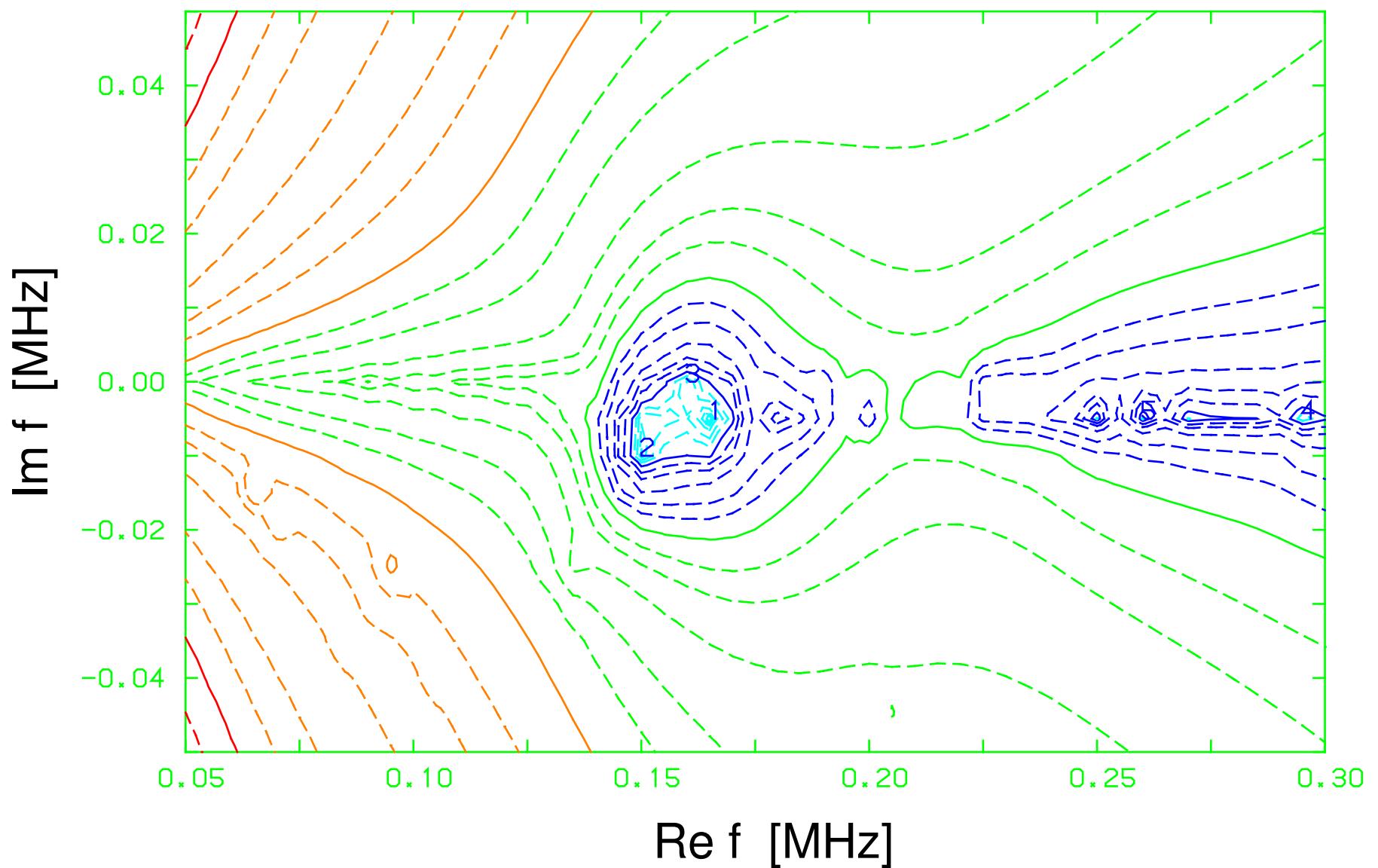
$R$	3.5016 m
$a$	0.9837 m
$\kappa$	.2810
$\delta$	0.3098
$b/a$	1.1
$B_0$	3.3119 T
$I_p$	1.6945 MA
$n_e(0)$	$0.2356 \cdot 10^{20} \text{ m}^{-3}$
$n_e(a)$	$0.05 \cdot 10^{20} \text{ m}^{-3}$
$T_e(0)$	4.1 keV
$T_e(a)$	0.8 keV
$T_D(0)$	3.7 keV
$T_D(a)$	0.4 keV

## Radial profile of Alfvén resonance frequency

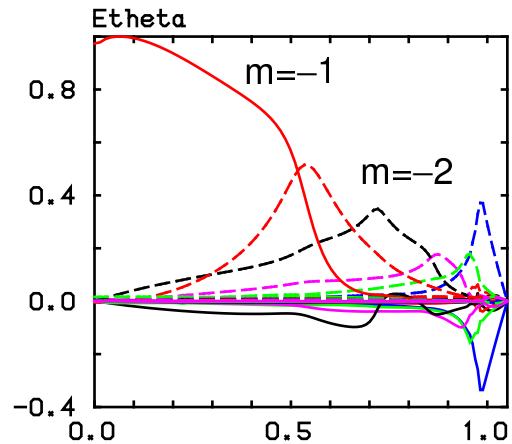


# Complex Eigen Frequency of Alfvén Eigenmode

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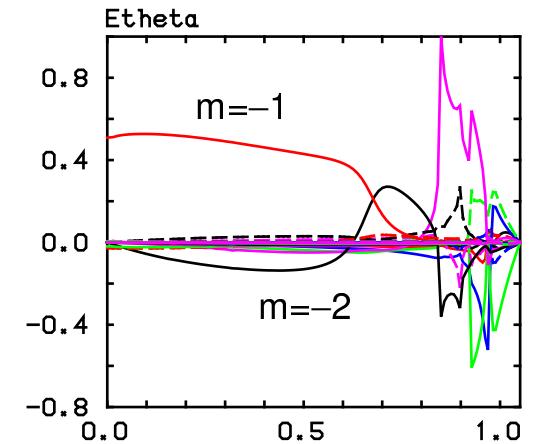


# Radial Mode Structure of Alfvén Eigenmode ( $n = 1$ )



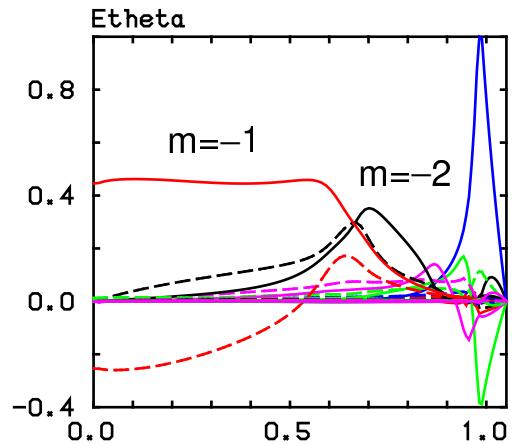
$f_r = 97.0 \text{ kHz}$

$f_i = -23.6 \text{ kHz}$



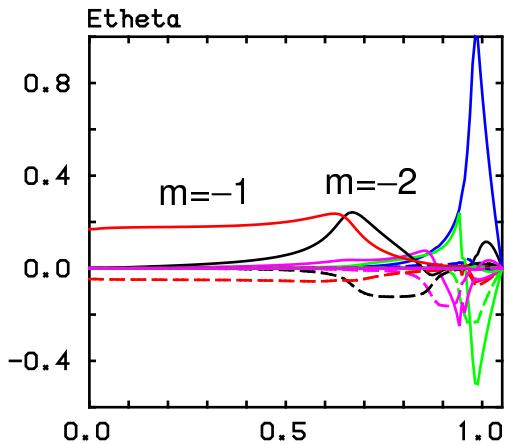
$f_r = 199.4 \text{ kHz}$

$f_i = -3.37 \text{ kHz}$



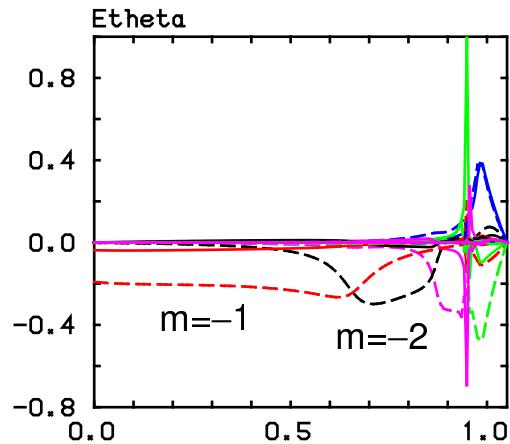
$f_r = 136.4 \text{ kHz}$

$f_i = -24.1 \text{ kHz}$



$f_r = 149.8 \text{ kHz}$

$f_i = -8.12 \text{ kHz}$

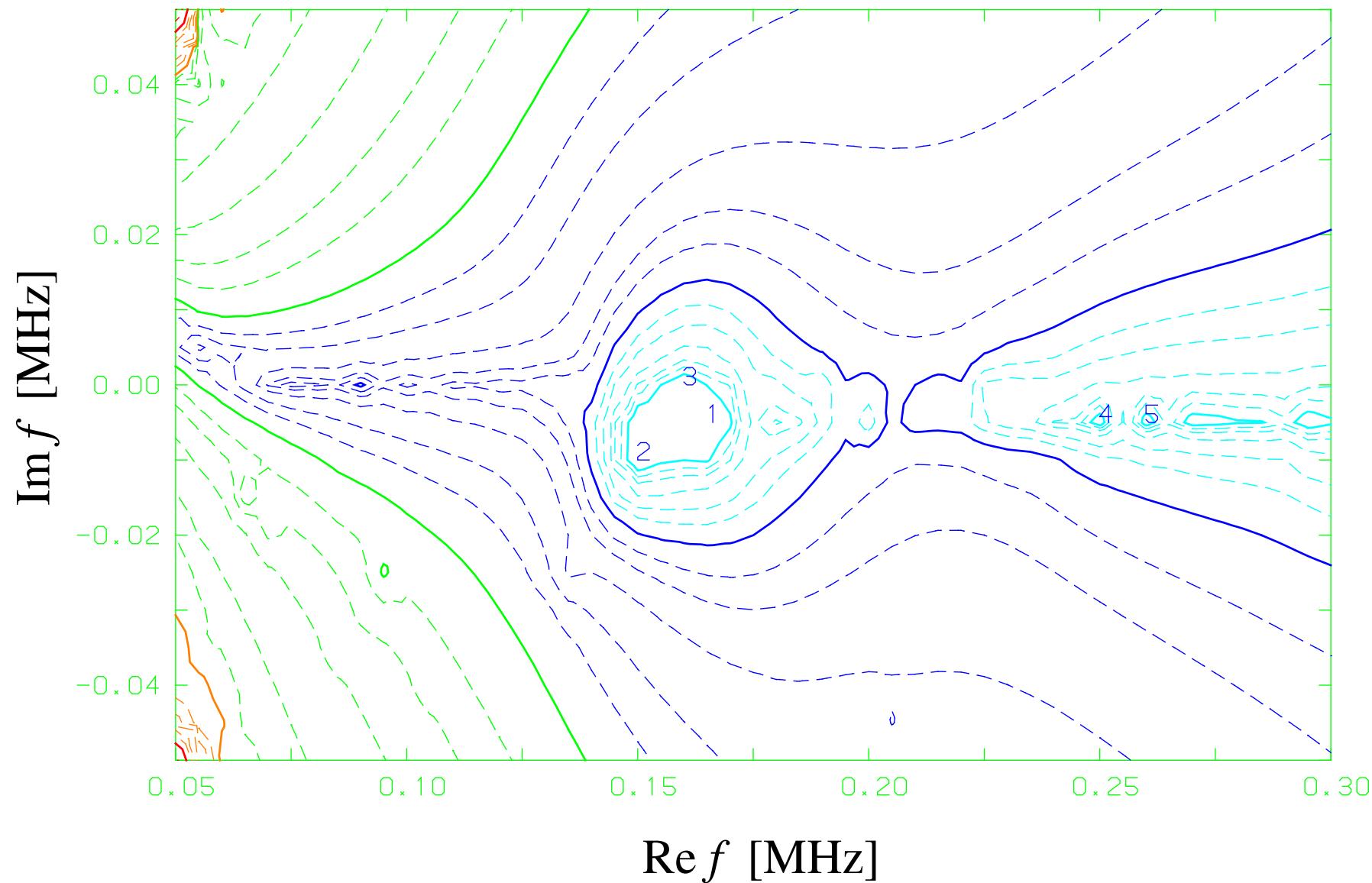


$f_r = 164.3 \text{ kHz}$

$f_i = -5.08 \text{ kHz}$

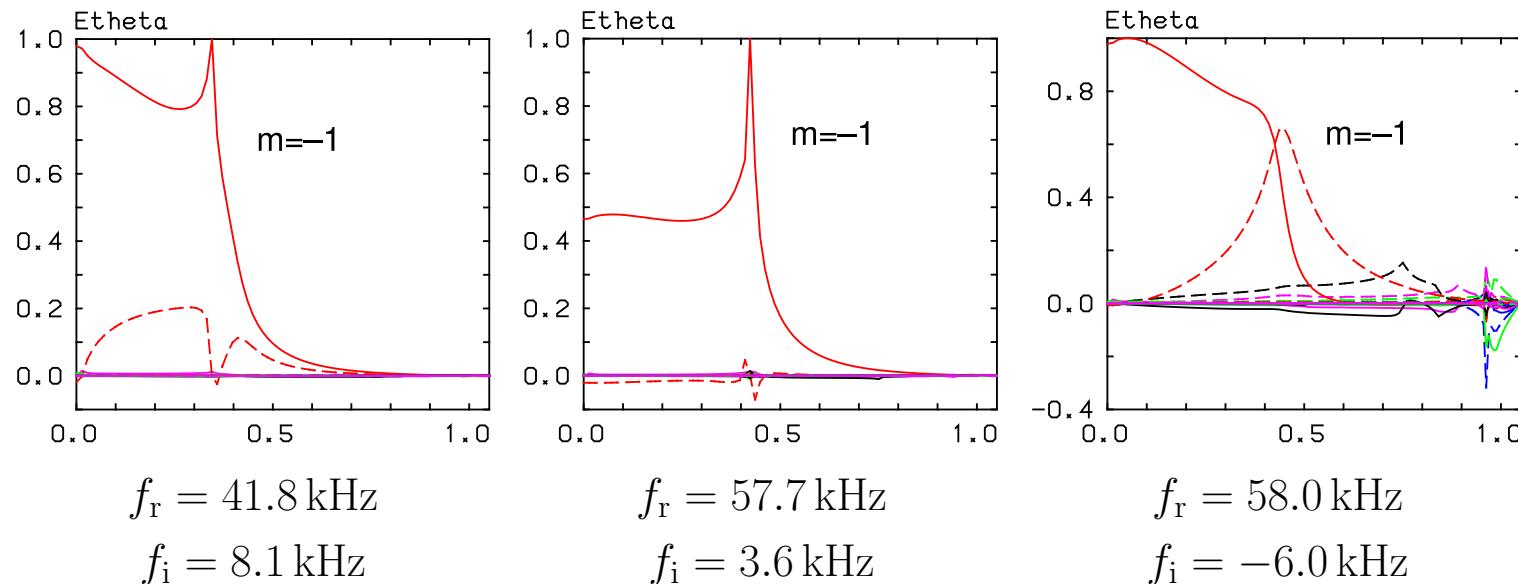
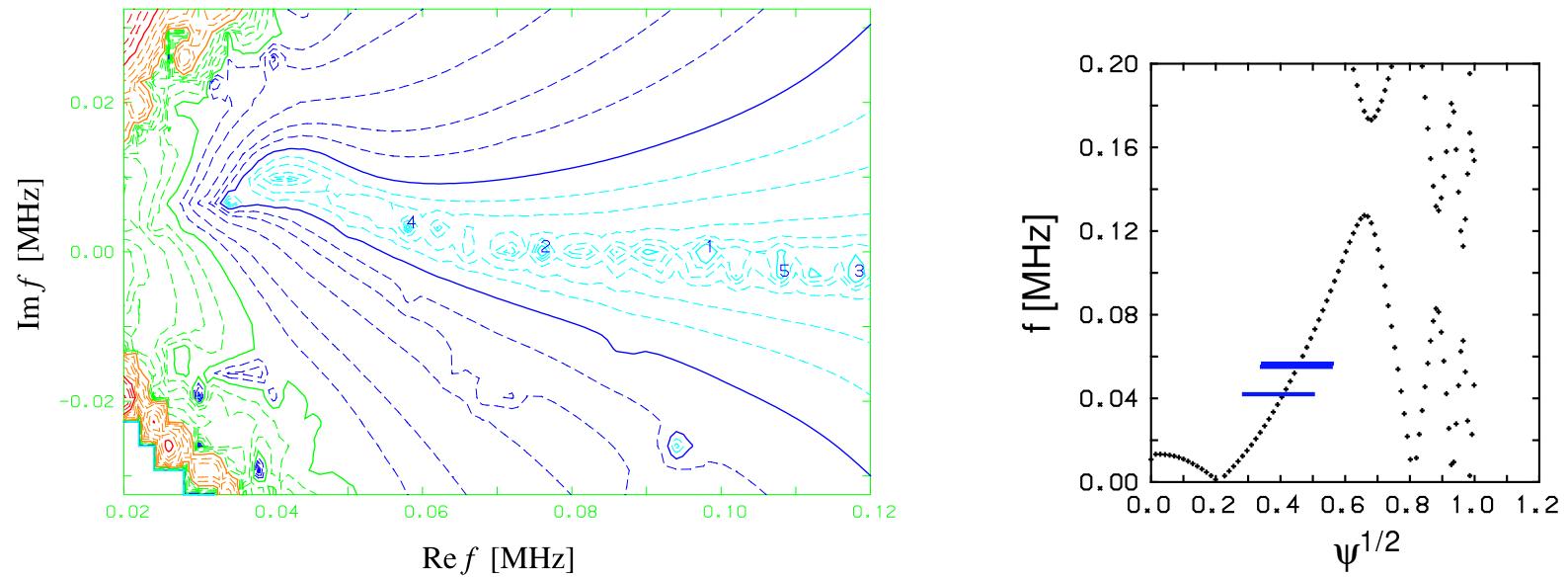
# Mode Structure with Energetic Particle

- $n_{F0} = 2 \times 10^{17} \text{ m}^{-3}$ ,  $T_B = 500 \text{ keV}$ ,  $L_{nB} = 0.5 \text{ m}$



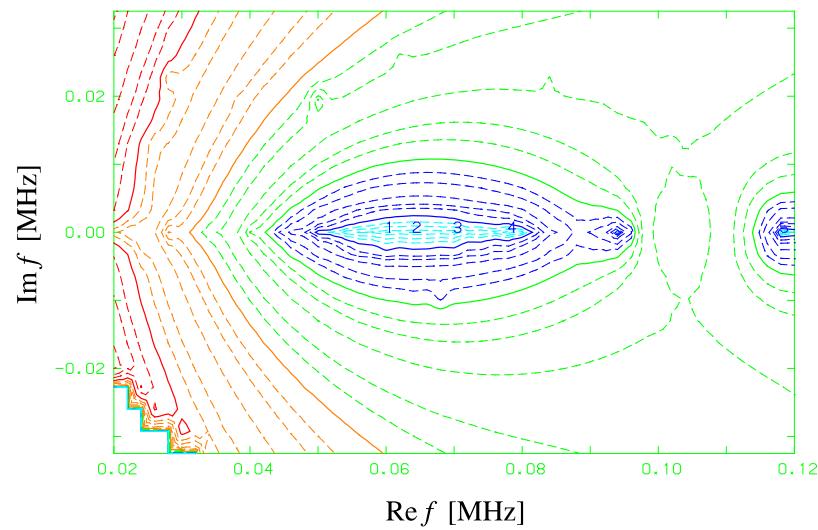
# Modes and Eigenfunctions Driven by Energetic Ions

- $n_{F0} = 2 \times 10^{17} \text{ m}^{-3}$ ,  $T_F = 500 \text{ keV}$ ,  $L_{nF} = 0.5 \text{ m}$ ,  $n = 1$

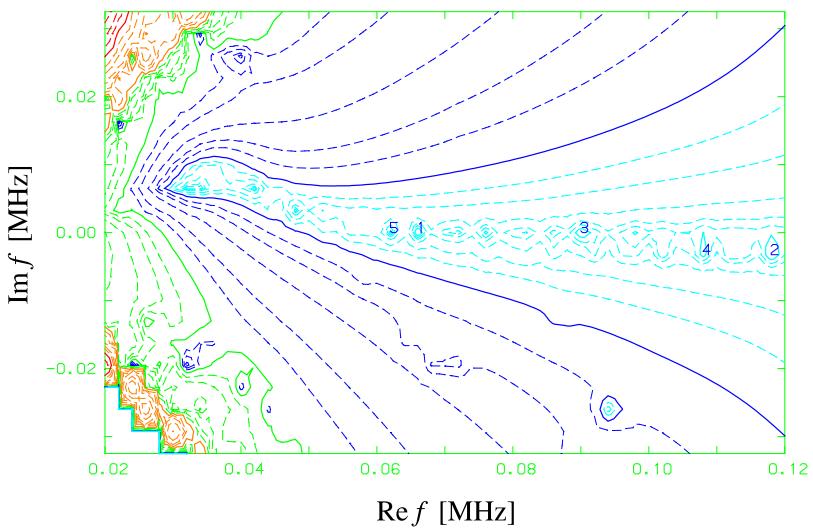


# Parameter Dependence of Mode Structure

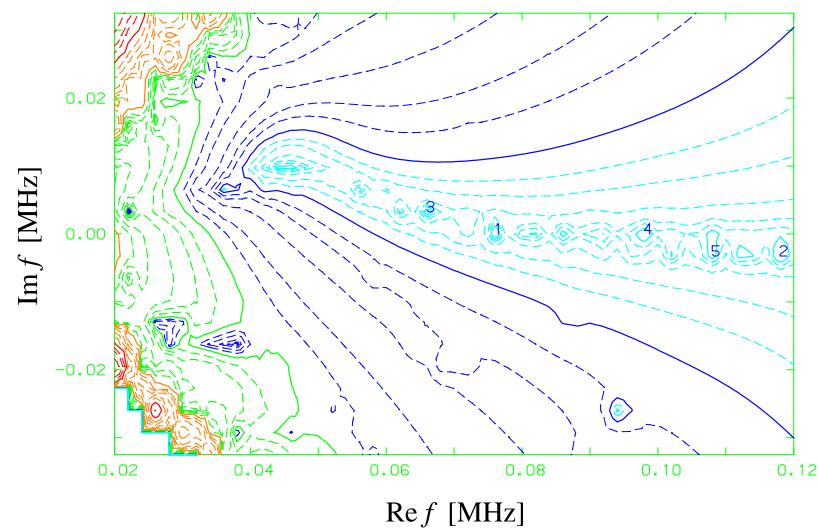
$$n_{F0} = 0 \times 10^{17} \text{ m}^{-3}, T_B = 0.5 \text{ MeV}$$



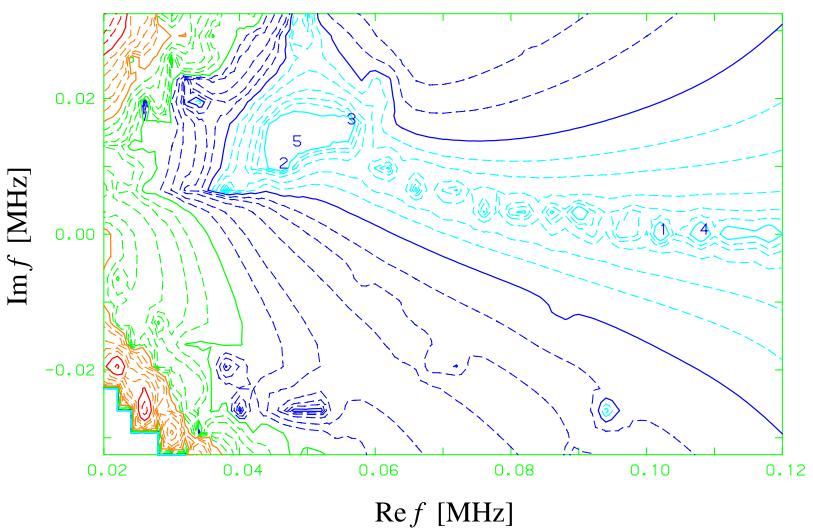
$$n_{F0} = 1 \times 10^{17} \text{ m}^{-3}, T_B = 0.5 \text{ MeV}$$



$$n_{F0} = 3 \times 10^{17} \text{ m}^{-3}, T_B = 0.5 \text{ MeV}$$

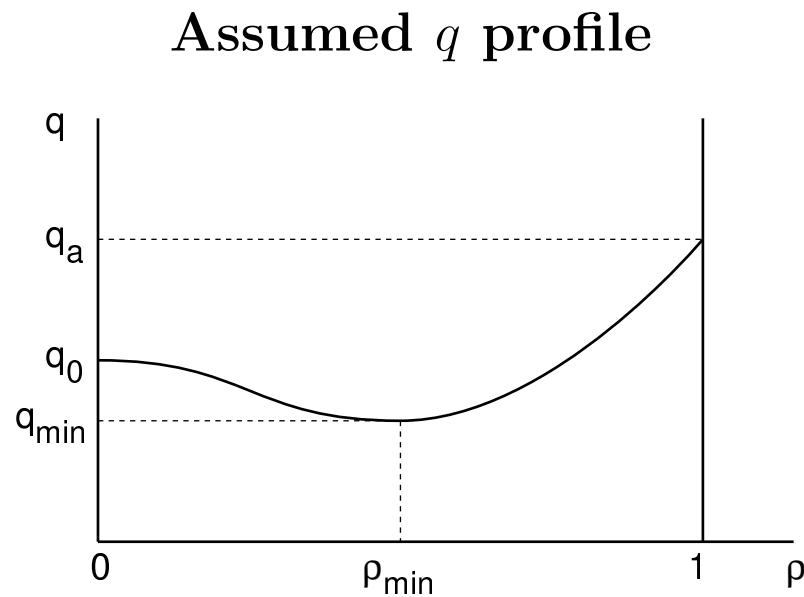


$$n_{F0} = 1 \times 10^{17} \text{ m}^{-3}, T_B = 1 \text{ MeV}$$



# Analysis of TAE in Reversed Shear Configuration

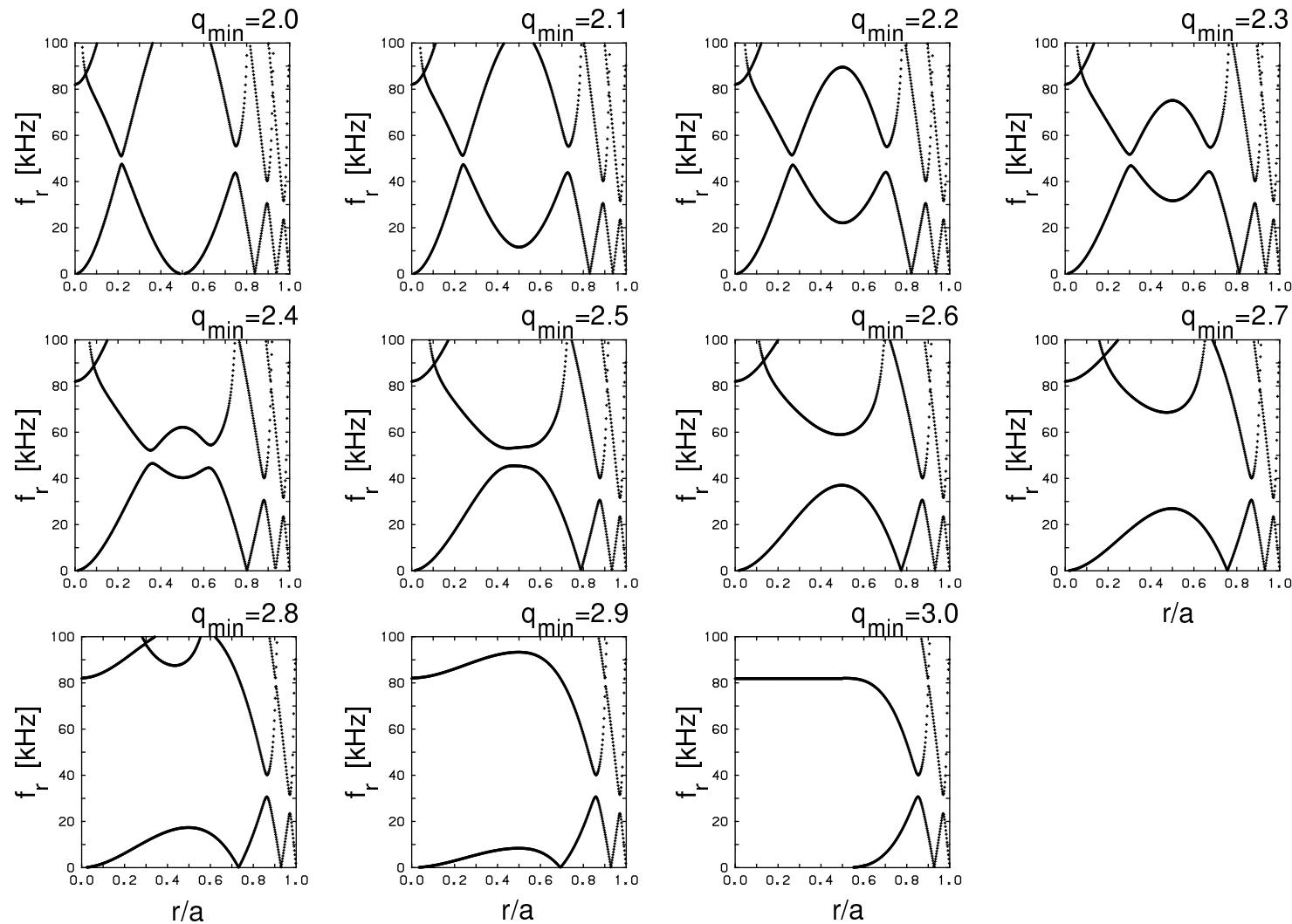
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## Plasma Parameters

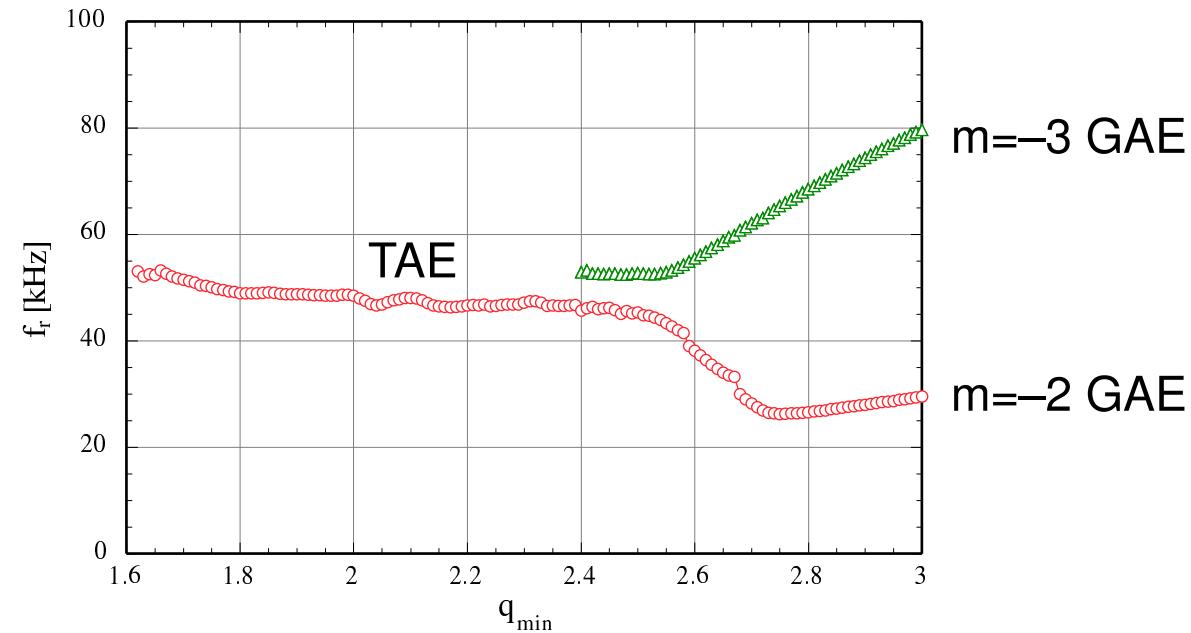
Major Radius	$R_0$	3 m
Minor Radius	$a$	1 m
Wall Radius	$b$	1.2 m
Toroidal Magnetic Field	$B_0$	3 T
Center Electron Density	$n_e(0)$	$10^{20} \text{ m}^{-3}$
Edge Electron Density	$n_e(a)$	$10^{20} \text{ m}^{-3}$
Central Temperature	$T(0)$	3 keV
Edge Temperature	$T(a)$	3 keV
Ion Species		Deuterium
Central Safety Factor	$q(0)$	3
Edge Safety Factor	$q(a)$	5
Toroidal Mode Number	$n$	1
$q$ -Minimum Radius	$\rho_{\min}$	0.5

# $q_{\min}$ Dependence of Alfvén Frequency Profile

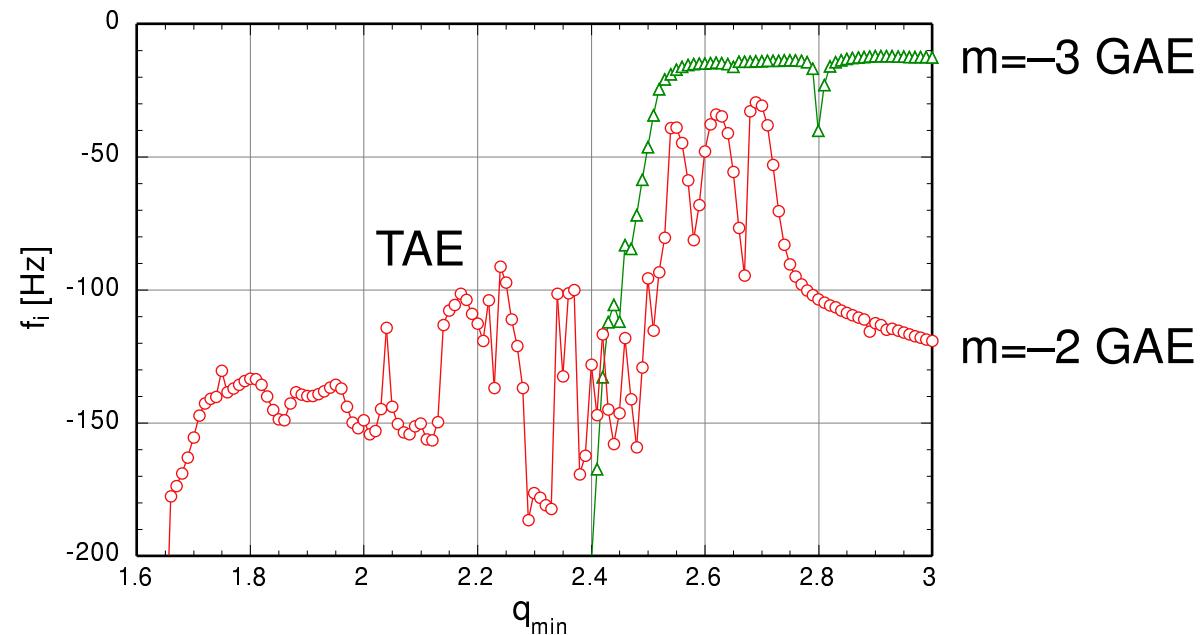


# $q_{\min}$ Dependence of Eigen Frequency and Damping Rate

Eigen Frequency

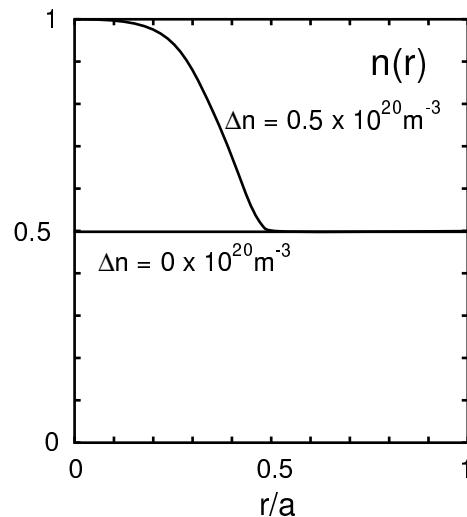


-Damping Rate

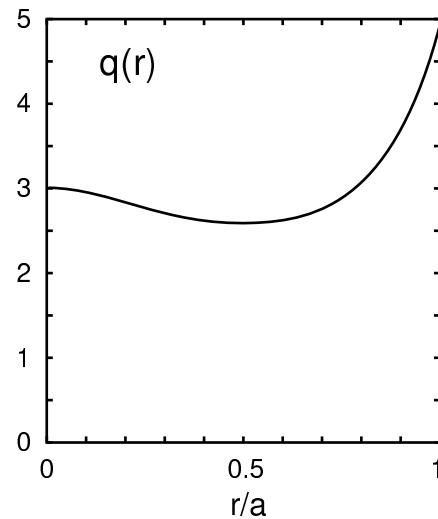


# Effect of ITB Density Profile on TAE

Density profile

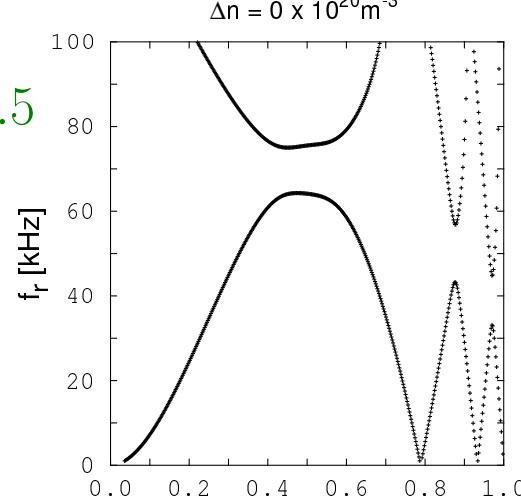


$q$  profile

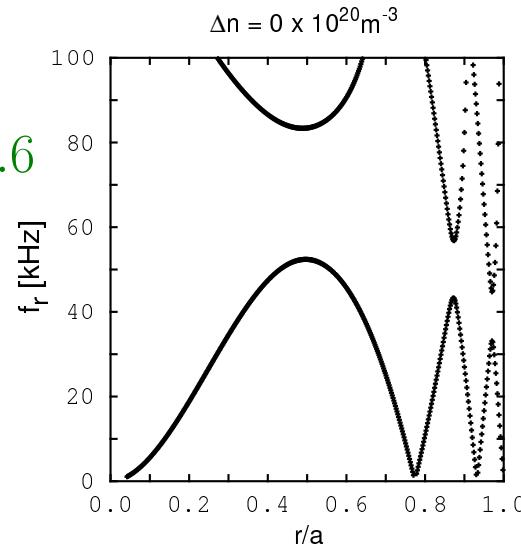


Alfvén resonance frequency

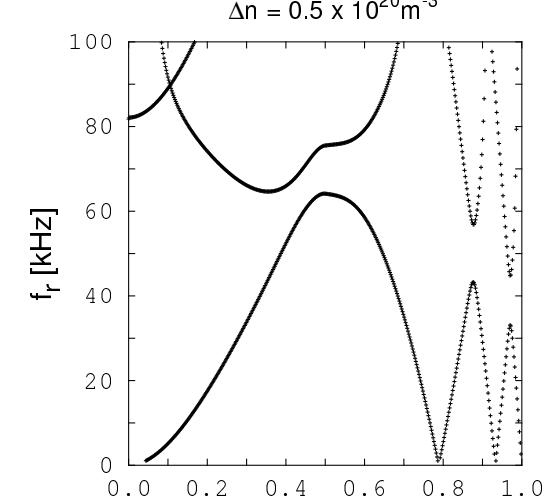
$$q_{\min} = 2.5$$



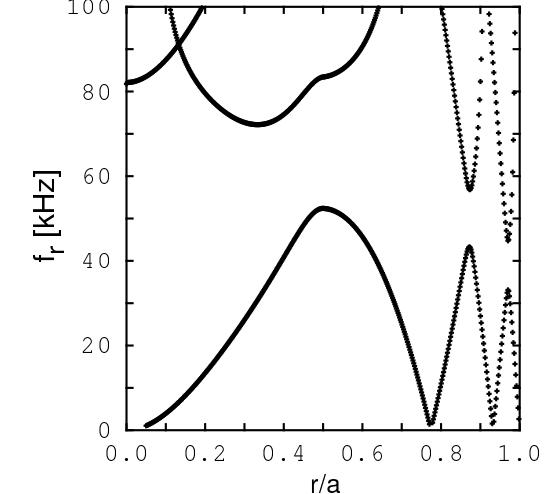
$$q_{\min} = 2.6$$



$$\Delta n = 0 \times 10^{20} \text{ m}^{-3}$$

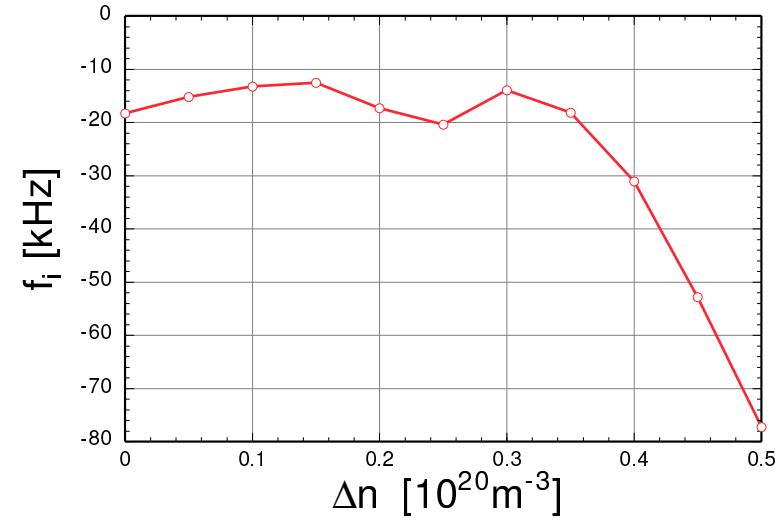
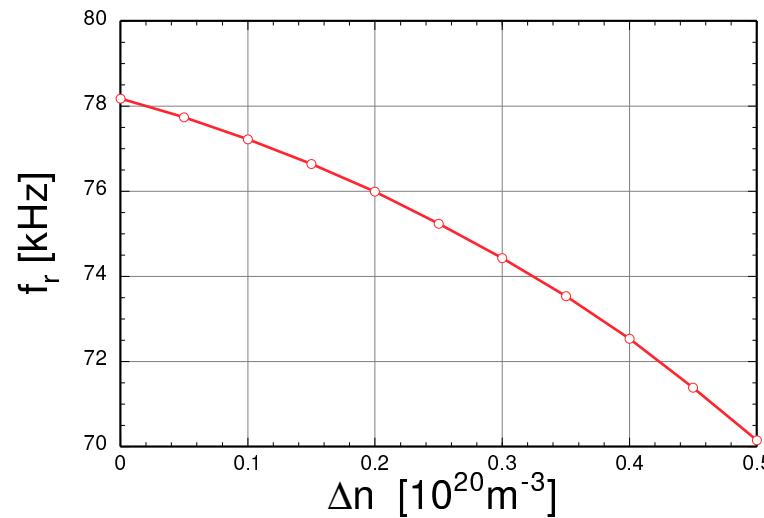


$$\Delta n = 0.5 \times 10^{20} \text{ m}^{-3}$$



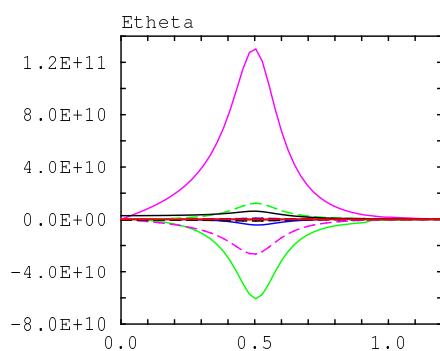
# Damping Rate and Eigenfunction

## Eigen frequency and damping rate

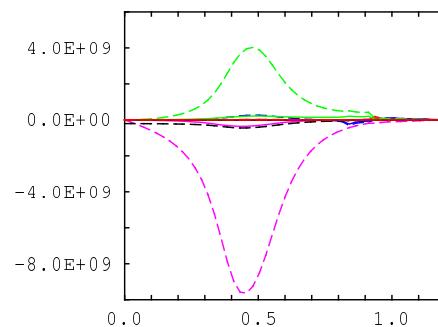


## Eigenfunction

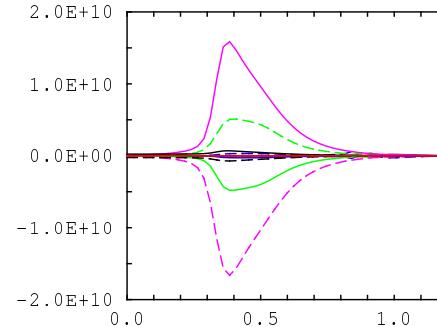
$\Delta n = 0.0$



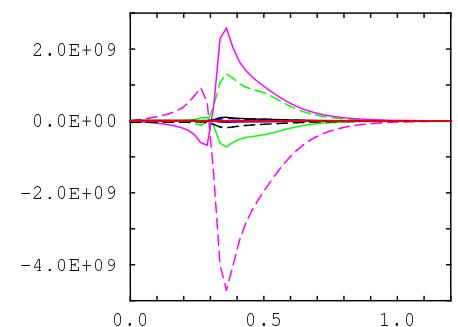
$\Delta n = 0.2$



$\Delta n = 0.4$



$\Delta n = 0.5$



## Summary

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- We studied ICRF waves in a toroidal helical plasma and Alfvén eigenmodes in tokamak plasmas using the 3D full wave code, TASK/WM.
- Characteristics of ICRF heating in LHD was studies using the cold plasma model. Dependence on the minority ion ratio agrees with experimental observation.
- The toroidal rotation changes the TAE frequency mainly through the change of gap position and  $q$  value.
- Destabilization by co-rotation agrees with experimental observation in JT-60U, though the stability is sensitive to the Alfvén resonance near the plasma surface.
- The mode structure of the EPM/RTAE below the gap frequency was studied. Two types of modes can be destabilized by the energetic ions; strongly damped TAE mode and weakly damped shear Alfvén mode.

- Reversed magnetic configuration supports GAE with single dominant poloidal mode number. The eigen frequency is close to the lower bound of the frequency gap and increases quickly with the decrease of  $q_{\min}$ . The density jump at ITB usually stabilizes TAEs.
- Future work
  - Kinetic analysis of ICRF heating in LHD
  - Systematic analysis of EPM/RTAE destabilized by energetic ions
  - Kinetic Analysis of low-frequency modes including the effect of particle orbit