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MHD Linear Stability Analysis Using a Full Wave Code

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Outlines

- Introduction
- TASK/WA code
  - Full Wave Analysis
  - Derivation of resistive MHD dielectric tensor
- Benchmark tests
- Resistive wall mode including the effect of ferromagnetism
  - Dispersion relation
  - Comparisons with TASK/WA code
  - Effect of plasma rotation
- Summary

Introduction

- **Various kinds of extended MHD models**
  - **Ideal MHD**
  - **Resistive MHD**
  - **Hall MHD**
  - **Finite-electron-mass MHD**
  - **Two-fluid MHD**
  - **Kinetic models**
    - **Uniform kinetic model**: $\omega \gg \omega_*, \omega_\mathrm{d}, \omega_\mathrm{B}$
      - No FLR, Reduced FLR, Differential OP, Integral OP, Spectral
    - **Gyrokinetic model**: $\omega \gg \omega_\mathrm{B}$
    - **Hamiltonian model**: three (adiabatic) constants of motion

Full Wave Analysis

- **Electric field $\boldsymbol{E}$ with a complex frequency ω**
- **Maxwell's equation and the dielectric tensor**

$$\nabla \times \frac{1}{\mu(\boldsymbol{r})} [\nabla \times \boldsymbol{E}(\boldsymbol{r})] - \omega^2 \epsilon_0 \int \mathrm{d}\boldsymbol{r}' \overleftrightarrow{\epsilon}(\boldsymbol{r}, \boldsymbol{r}') \cdot \boldsymbol{E}(\boldsymbol{r}') = \mathrm{i}\,\omega \boldsymbol{j}_{\mathrm{ext}} \upsilon r)$$

- **Type of analyses**
  - **Antenna excitation**:
    - ω: real (boundary-value problem)
    - Analysis of wave heating and current drive
    - Loading impedance analysis (weakly stable eigenmode)
  - **Spontaneous excitation**:
    - ω: complex (eigen-value problem)
    - Analysis of linear stability

Previous Analysis

- **TASK/WM**
  - **Fourier mode expansion in poloidal and toroidal directions**
  - **Maxwell's equation and the dielectric tensor**

$$\nabla\times\frac{1}{\mu_0}\left[\nabla\times\boldsymbol{E}_{mn}(\rho)\right]-\omega^2\epsilon_0\sum_{m'n'}\overleftrightarrow{\epsilon}_{m-m',n-n'}(\rho)\cdot\boldsymbol{E}_{m'n'}(\rho) = \mathrm{i}\,\omega\boldsymbol{j}_{\mathrm{ext}}(\boldsymbol{r})$$

  - **(Uniform kinetic + gyrokinetic) dielectric tensor**
    - Plasma dispersion function for parallel interaction
    - Electron finite mass
    - No finite Larmor radius effects
  - **Finite difference method in radial direction**
  - **Alfvén eigenmode analysis**
  - **Excitation by energetic particle**
  - **No coupling with drift waves**

Present Analysis

- **TASK/WA**
  - Fourier mode expansion in poloidal and toroidal directions
  - Maxwell's equation and the dielectric tensor

$$\nabla\times\frac{1}{\mu(\rho)}\left[\nabla\times \boldsymbol{E}_{mn}(\rho)\right]-\omega^2\epsilon_0\sum_{m'n'}\overleftrightarrow{\epsilon}_{m-m',n-n'}(\rho)\cdot\boldsymbol{E}_{m'n'}(\rho) = \mathrm{i}\,\omega\boldsymbol{j}_{\mathrm{ext}}(\boldsymbol{r})$$

  - Resistive MHD dielectric tensor
  - Finite element method in radial direction
  - Higher numerical accuracy
  - Comparison with previous Resistive MHD analyses
  - Analysis of Resistive Wall Mode (RWM)

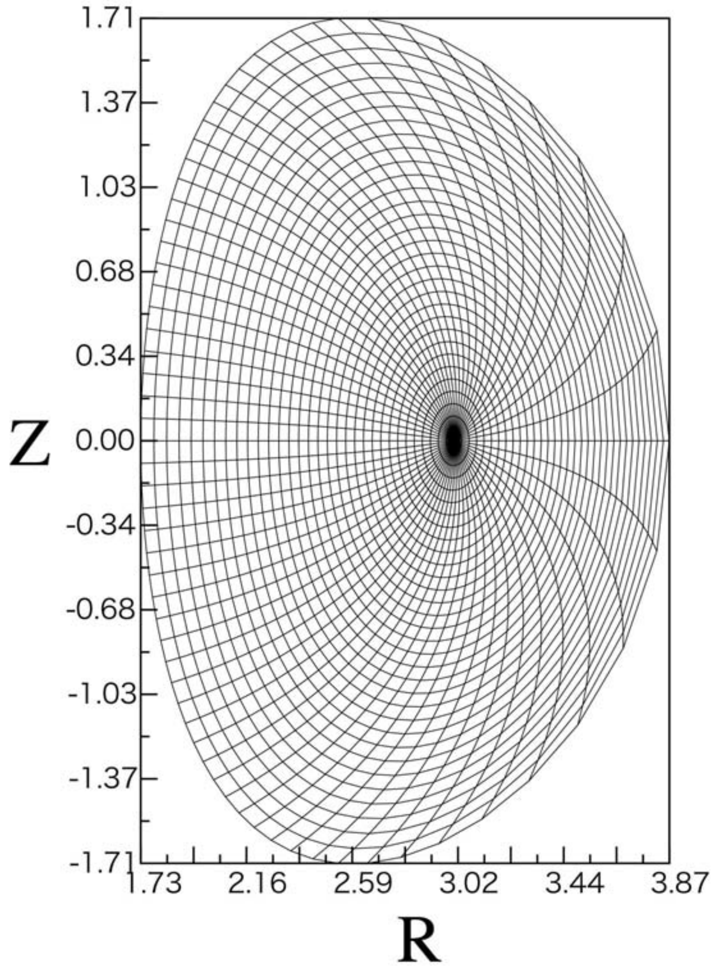
TASK/WA: Coordinates

- Flux coordinate: $(s, \Theta, \varphi) = (x^1, x^2, x^3)$
  - φ: Toroidal angle
  - $s = \sqrt{\psi_\mathrm{p}}$: Radial coordinate
  - Θ: Poloidal angle (Boozer coordinate)

— $k_{||}R = n\dfrac{B_\varphi}{B} + \dfrac{mR}{r}\dfrac{B_\Theta}{B}$: constant

— $k_{||} = \dfrac{n}{R}\dfrac{B_\varphi}{B} + \dfrac{m}{r}\dfrac{B_\Theta}{B}$: not constant

— $k_{||} = 0$ on a rational surface



TASK/WA: Finite Element Method

- **Weak form of Maxwell's equation**: ($\boldsymbol{w}$: weighting function)

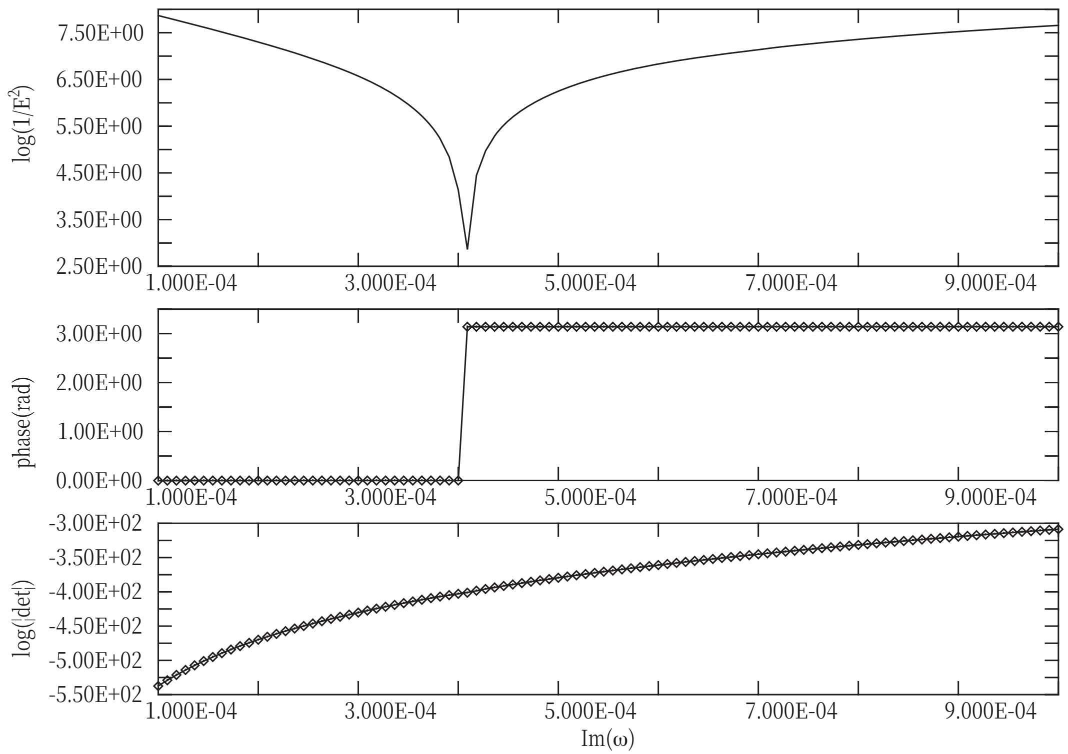
$$\int (\nabla \times \boldsymbol{w}) \cdot \frac{1}{\mu} (\nabla \times \boldsymbol{E})\, dV + \omega^2 \epsilon_0 \int \boldsymbol{w} \cdot \overleftrightarrow{\epsilon} \cdot \boldsymbol{E}\, dV + \mathrm{i}\,\omega \int \boldsymbol{w} \cdot \boldsymbol{j}_{\mathrm{ext}}\, dV$$

- **Interpolation function**
  - **In plasma**: second-order Lagrange polynomial
  - **In vacuum**: linear function
- **Boundary conditions**
  - **On magnetic axis**:
    - — $E_2 \approx sE_\theta \to 0$ for $s \to 0$
    - — Finiteness of $\mathrm{i}\,\omega \boldsymbol{B} = \nabla \times \boldsymbol{E}$
  - **On plasma surface**:
    - — Surface term due to a derivative of discontinuous parameters.
  - **On perfectly conducting surface**: $E_2 = 0, \quad E_3 = 0$

TASK/WA: Eigenmode analysis

- **Excitation, $j_{\rm ext}$, proportional to the electron density**
- **Find complex ω which maximize the volume-integrated electric field amplitude**

Inverse of volume-integrate wave amplitude, phase and amplitude of the determinant of coefficient matrix as a function of the growth rate



Resistive MHD Dielectric Tensor

- **Resistive MHD equation**

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} + \Gamma p \nabla \cdot \boldsymbol{u} = 0$$

$$\rho_m \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + \nabla p = \boldsymbol{j} \times \boldsymbol{B}$$

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = \eta \boldsymbol{j}$$

- **Linearization**: $X = X_0 + X_1$
- **Laplace Transform**: complex ω
- **Flux coordinate**: (s, Θ, φ)=(x^1, x^2, x^3)
    - Co-variant and contra-variant expression

Matrix Expression (I)

- Definitions

$$\bar{\boldsymbol{u}}_0 = (0, u_0^2, u_0^3), \qquad \bar{\boldsymbol{u}}_1 = (u_1^1, u_1^2, u_1^3)$$

$$\bar{\boldsymbol{B}}_0 = (0, B_0^2, B_0^3), \qquad \bar{\boldsymbol{E}}_1 = (E_{1,1}, E_{1,2}, E_{1,3})$$

$$\overleftrightarrow{g} = \{g^{ij}\}, \qquad \bar{\boldsymbol{\nabla}} = (\mathrm{d}/\mathrm{d}x^1, \mathrm{d}/\mathrm{d}x^2, \mathrm{d}/\mathrm{d}x^3)$$

$$u_0^i u_1^j \Lambda_{ij}^{\alpha} \equiv \bar{\boldsymbol{u}}_0 \cdot \overleftrightarrow{\Lambda} \cdot \bar{\boldsymbol{u}}_1, \qquad \bar{\boldsymbol{u}}_1 \cdot \bar{\boldsymbol{\nabla}} \bar{\boldsymbol{u}}_0 + 2\bar{\boldsymbol{u}}_0 \cdot \overleftrightarrow{\Lambda} \cdot \bar{\boldsymbol{u}}_1 \equiv \overleftrightarrow{U} \cdot \bar{\boldsymbol{u}}_1$$

$$\bar{\boldsymbol{j}}_1 \times \bar{\boldsymbol{B}}_0 \equiv B_{\mathrm{axis}} \overleftrightarrow{R} \cdot \bar{\boldsymbol{j}}_1$$

$$\frac{1}{\omega}\left(\bar{\boldsymbol{k}}_e \bar{\boldsymbol{j}}_0 - \bar{\boldsymbol{k}}_e \cdot \bar{\boldsymbol{j}}_0 \overleftrightarrow{I}\right) \equiv \overleftrightarrow{K}(\bar{\boldsymbol{j}}_0), \qquad \omega' = \omega - \bar{\boldsymbol{u}}_0 \cdot \bar{\boldsymbol{k}}$$

$\bar{\boldsymbol{k}} = (-\mathrm{i}\,\mathrm{d}/\mathrm{d}x^1, m, n)$: wave number

Matrix expression (II)

- **Adiabatic equation of state**

$$\mathrm{i}\,\omega' p_1 = \frac{\mathrm{d}p_0}{\mathrm{d}x^1} u_1^1 + \Gamma p_0 \boldsymbol{\nabla} \cdot \boldsymbol{u}_1$$

- **Equation of motion**

$$\rho_0 \left(-\mathrm{i}\,\omega' + \overleftrightarrow{U} \right) \cdot \bar{\boldsymbol{u}}_1 + \overleftrightarrow{g} \cdot \bar{\boldsymbol{\nabla}} p_1 = J B_{\mathrm{axis}} \overleftrightarrow{g} \cdot \overleftrightarrow{R} \cdot \bar{\boldsymbol{j}}_1 + \overleftrightarrow{g} \cdot \overleftrightarrow{K}(\bar{\boldsymbol{j}}_0) \cdot \bar{\boldsymbol{E}}_1$$

- **Ohm's law**

$$\overleftrightarrow{g} \cdot \bar{\boldsymbol{E}}_1 + J B_{\mathrm{axis}} \overleftrightarrow{g} \cdot \overleftrightarrow{R} \cdot \bar{\boldsymbol{u}}_1 + \overleftrightarrow{g} \cdot \overleftrightarrow{K}(\bar{\boldsymbol{u}}_0) \cdot \bar{\boldsymbol{E}}_1 = \eta \bar{\boldsymbol{j}}_1$$

- **Finally we obtain the linear relation between $\bar{u}_1$ and $\bar{E}_1$**

$$\left[\eta \rho_0 \left(-\mathrm{i}\,\omega' + \overleftrightarrow{U} \right) - (J B_{\mathrm{axis}})^2 \overleftrightarrow{g} \cdot \overleftrightarrow{R} \cdot \overleftrightarrow{g} \cdot \overleftrightarrow{R} \right] \cdot \bar{\boldsymbol{u}}_1 + \frac{\eta}{\mathrm{i}\,\omega'} \overleftrightarrow{g} \cdot \bar{\boldsymbol{\nabla}} \left(\frac{\mathrm{d}p_0}{\mathrm{d}x^1} u_1^1 + \Gamma p_0 \boldsymbol{\nabla} \cdot \boldsymbol{u}_1 \right)$$

$$= \left[J B_{\mathrm{axis}} \overleftrightarrow{g} \cdot \overleftrightarrow{R} \cdot \overleftrightarrow{g} + J B_{\mathrm{axis}} \overleftrightarrow{g} \cdot \overleftrightarrow{R} \cdot \overleftrightarrow{g} \cdot \overleftrightarrow{K}(\bar{\boldsymbol{u}}_0) + \eta \overleftrightarrow{g} \cdot \overleftrightarrow{K}(\bar{\boldsymbol{j}}_0) \right] \cdot \bar{\boldsymbol{E}}_1$$

Dielectric Tensor

- **Induced current**: $\boldsymbol{J}_1 = \overleftrightarrow{\sigma} \cdot \boldsymbol{E}_1$
- **Electrical conductivity tensor**

$$\overleftrightarrow{\sigma} = \overleftrightarrow{\sigma}_0 + \overleftrightarrow{\sigma}_x \frac{\mathrm{d}}{\mathrm{d}x^1}$$

$$+ \overleftrightarrow{\sigma}_1 \cdot \frac{\mathrm{d}}{\mathrm{d}x^1} \overleftrightarrow{R}_0 + \overleftrightarrow{\sigma}_1 \cdot \frac{\mathrm{d}}{\mathrm{d}x^1} \overleftrightarrow{R}_1 \frac{\mathrm{d}}{\mathrm{d}x^1}$$

$$+ \overleftrightarrow{\sigma}_2 \cdot \frac{\mathrm{d}}{\mathrm{d}x^1} \overleftrightarrow{L}_{22} \cdot \frac{\mathrm{d}}{\mathrm{d}x^1} \overleftrightarrow{R}_0 + \overleftrightarrow{\sigma}_2 \cdot \frac{\mathrm{d}}{\mathrm{d}x^1} \overleftrightarrow{L}_{22} \cdot \frac{\mathrm{d}}{\mathrm{d}x^1} \overleftrightarrow{R}_1 \frac{\mathrm{d}}{\mathrm{d}x^1}$$

where $\overleftrightarrow{\sigma}_0$ and $\overleftrightarrow{\sigma}_x$ are 3×3 matrixes, $\overleftrightarrow{\sigma}_1$ and $\overleftrightarrow{\sigma}_2$ are 3×2 matrixes, $\overleftrightarrow{R}_0$ and $\overleftrightarrow{R}_1$ are 2×3 matrixes, $\overleftrightarrow{L}_{22}$ is a 2×2 matrix.

- **Dielectric tensor**

$$\overleftrightarrow{\epsilon} = \overleftrightarrow{I} + \frac{\mathrm{i}}{\omega\epsilon_0} \sum_s \overleftrightarrow{\sigma}_s$$

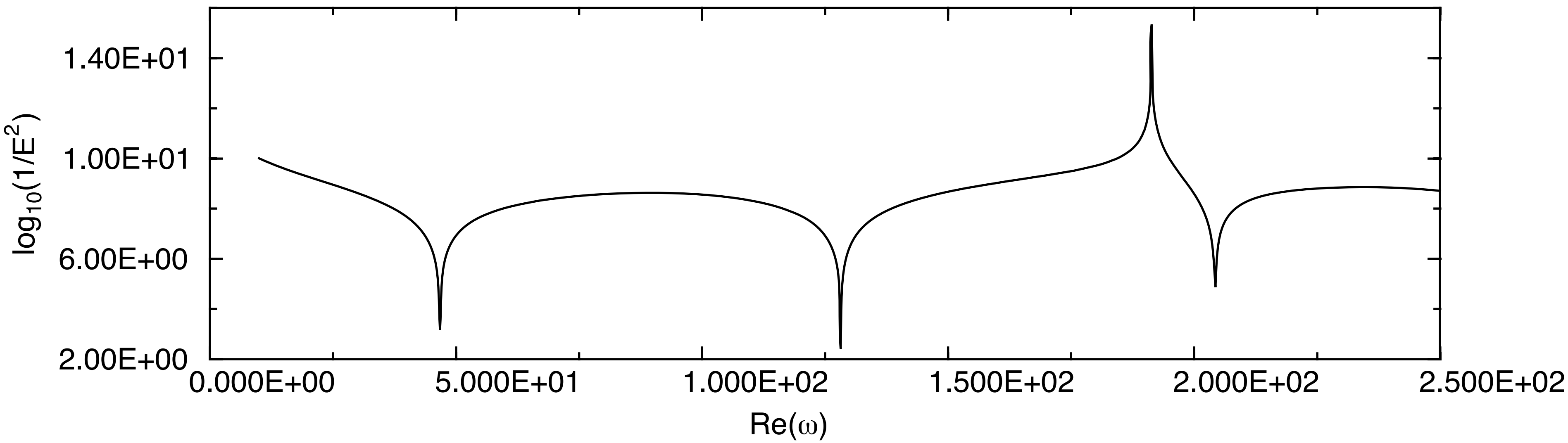
Circular Waveguide

- **Analytic solution**

$$k_z^2 + \frac{(j'_{ml})^2}{a^2} = \frac{\omega^2}{c^2}$$

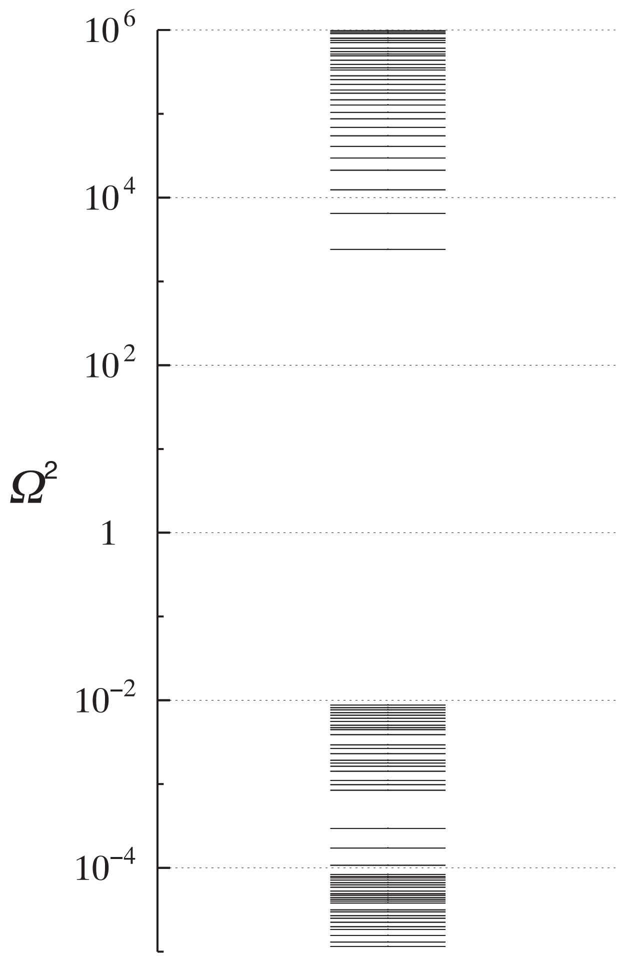
- 46.7MHz, 128.2MHz, 204.3MHz for $k_z = 1/3\,\mathrm{m}^{-1}$, $a = 2\,\mathrm{m}$C$m = 1$

Inverse of volume-integrated electric field amplitude



Spectrum in a Cylindrical Plasma

- **Square of frequency normalized by the Alfvén time $\tau_\mathrm{A} = R_0/v_\mathrm{A}$, $\Omega^2 = (\omega\tau_\mathrm{A})^2$**
- **From high frequency**: fast wave, shear Alfvén wave, slow wave
- **Good agreement with Fig. 5 in M. Chance, et. al, *Nucl. Fusion* 17 (1977) 65**
- Parameters
  - Uniform density, B_z, and q
  - Poloidal mode number $m = 2$
  - $nq = 1.9$



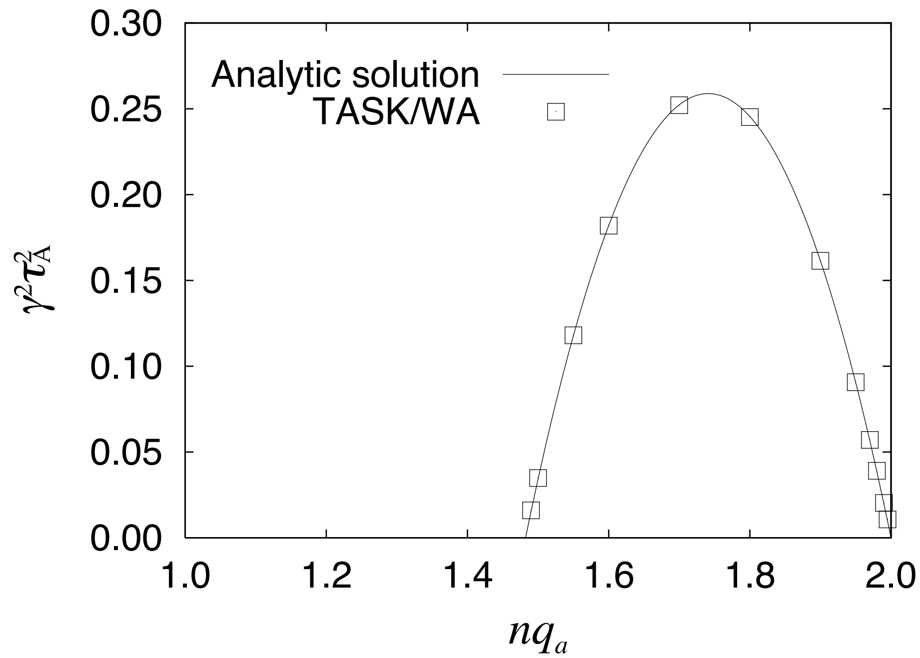
External Kink Mode in a Cylindrical Plasma

- **Flat current profile**
- **Analytic solution**

$$\gamma^2 q_a^2 \tau_{\mathrm{A}}^2 = 2(m - nq_a)\left[1 - \frac{m - nq_a}{1 - (a/b)^{2m}}\right]$$

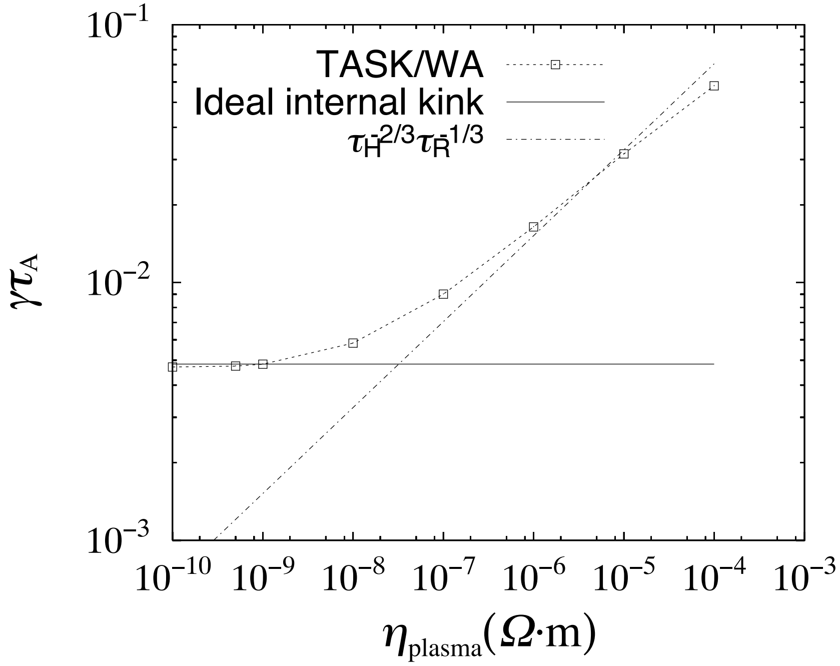
- $a = 1\,\mathrm{m}$, $b = 1.2a$C$L = 2\pi R_0 = 40\pi\,\mathrm{m}Cm = 2Cn = 1$

Growth rate as a function of nq

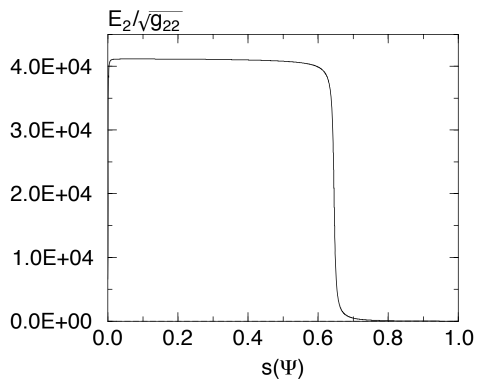


m = 1 Internal Kink Mode and Tearing Mode

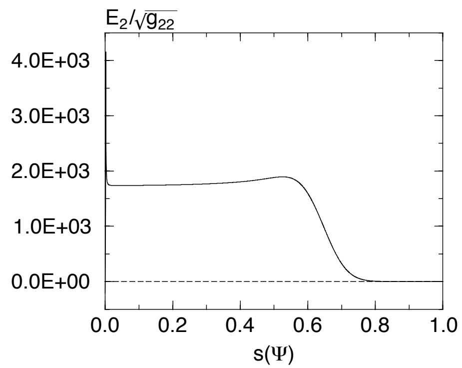
Growth rate as a function of plasma resistivity



$\eta = 5 \times 10^{-10}\Omega\cdot m$



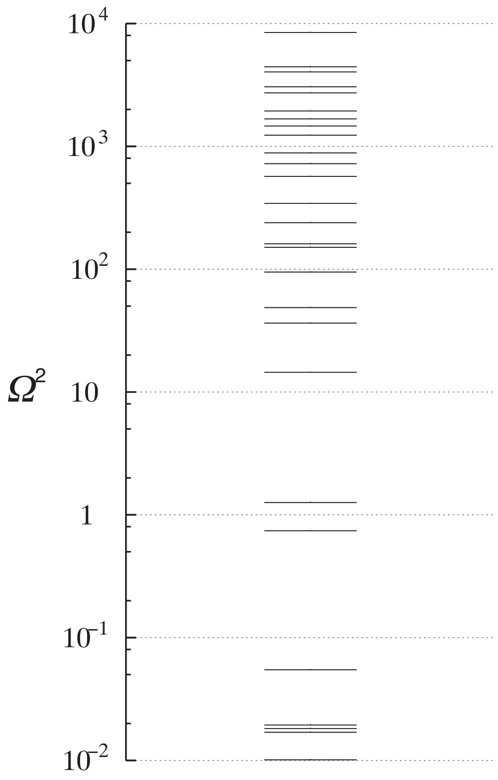
$\eta = 1 \times 10^{-4}\Omega\cdot m$



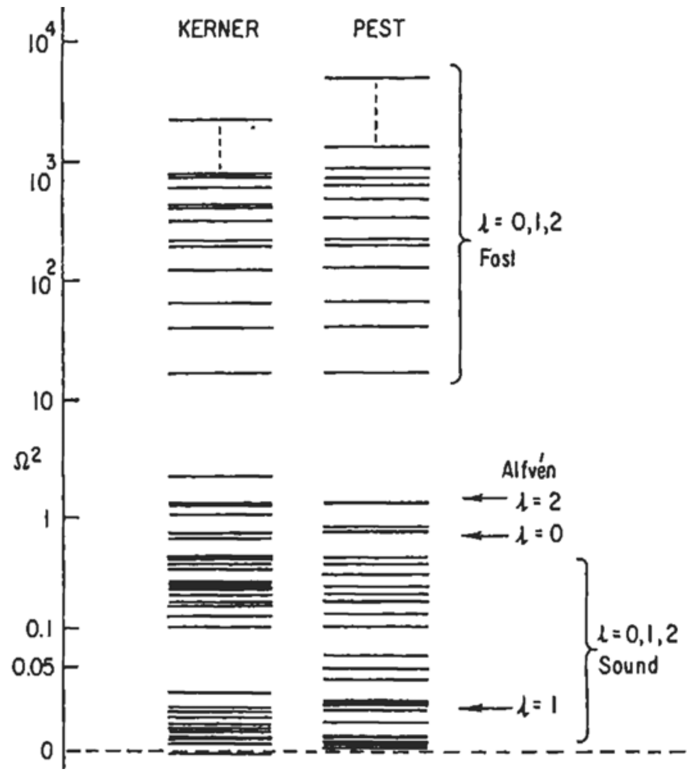
Spectrum in a Tokamak Plasma

- **Spectrum of Incompressible plasma in a Solovev equilibrium**
- **cf. Fig. 2 in M. Chance, *et. al*, *J. Comput. Phys* 28 (1978) 1**
- $\epsilon = a/R_0 = 1/20$, $\kappa = 1$, $q_0 = 0.08591$, $n = 10$, $m = 0, 1, 2$

TASK/WA

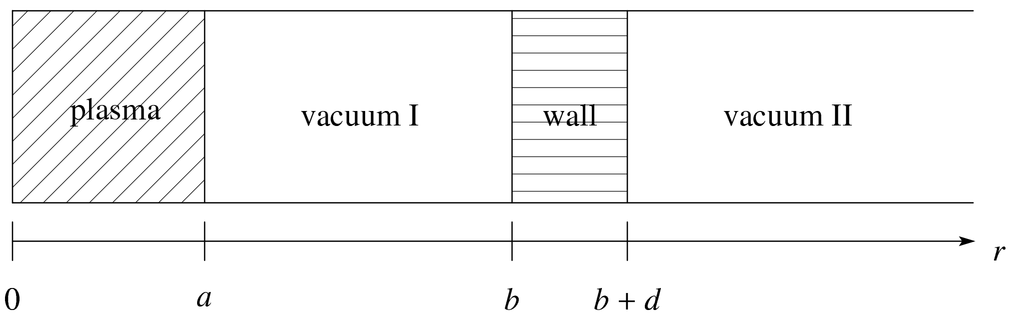


Chance et al.



Dispersion Relation of RWM

- **Cylindrical plasma:** (radius: a)
  - Axial length: $2\pi R_0$ (periodic)
  - Poloidal mode number: m
  - Toroidal mode number: n, $k = -n/R_0$, $(k^2 r^2 \ll 1)$
- **Surrounded by a resistive and ferromagnetic wall:** (radius: b)



  - Wall thickness: d
  - Wall permeability: μ_W, $(\hat{\mu} \equiv \mu_W/\mu_0)$
  - Wall resistivity: η_W

Dispersion Relation of RWM (I)

- **Response of the vacuum regions and the wall**

$$\alpha = \frac{\left[\lambda^2(b+d) - \dfrac{m^2\hat{\mu}^2}{b}\right](1-\mathrm{e}^{-2d\lambda}) - \lambda m\hat{\mu}(1+\mathrm{e}^{-2d\lambda})\dfrac{d}{b}}{\left[\lambda^2(b+d) + \dfrac{m^2\hat{\mu}^2}{b}\right](1-\mathrm{e}^{-2d\lambda}) + \lambda m\hat{\mu}(1+\mathrm{e}^{-2d\lambda})\left(2+\dfrac{d}{b}\right)}\left(\frac{a}{b}\right)^{2m}$$

where

$$\lambda^2 = \frac{\hat{\mu}\gamma\tau_{\mathrm{W}}}{bd} + \frac{m^2}{b^2} + k^2$$

- **Response of the plasma**

$$\alpha = 1 + \frac{2m}{(nq_a+m)/(nq_a-m) + \Delta_a(\gamma)(1+\hat{\gamma}^2) + \hat{\gamma}^2 - m}$$

where the normalized growth rate is

$$\hat{\gamma} = \gamma\tau_{\mathrm{A}}q_a/(m-nq_a)$$

Dispersion Relation of RWM (II)

- **Value of $\Delta_a(\gamma)$**
  - **When the q profile is flat**: $\Delta_a(\gamma) = m - 1$
  - **When the rational surface $q(r_{mn}) = m/n$ is close to the plasma surface, $r_{mn} \gtrsim a$**:

$$\Delta_a(\gamma) = \frac{\Delta_a(0)(1+\hat{\gamma}^2)^{-1}}{1 - \Delta_a(0)\left(1 - \int_1^\infty (y^2+\hat{\gamma}^2)^{-1}\mathrm{d}y\right)}, \qquad \Delta_a(0) \equiv \left(\frac{r\xi'}{\xi}\right)_a$$

where ξ is a solution of the Euler equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(f\frac{\mathrm{d}\xi}{\mathrm{d}r}\right) - g\xi = 0$$

$$f = \frac{rF^2}{k_0^2}, \qquad g = 2\frac{k^2}{k_0^2}(\mu_0 p)' + \left(\frac{k_0^2 r^2 - 1}{k_0^2 r^2}\right) rF^2 + 2\frac{k^2}{rk_0^4}\hat{F}F$$

$$F = kB_z + \frac{m}{r}B_\theta, \qquad k_0^2 = k^2 + \frac{m^2}{r^2}, \qquad \hat{F} = kB_z - \frac{m}{r}B_\theta$$

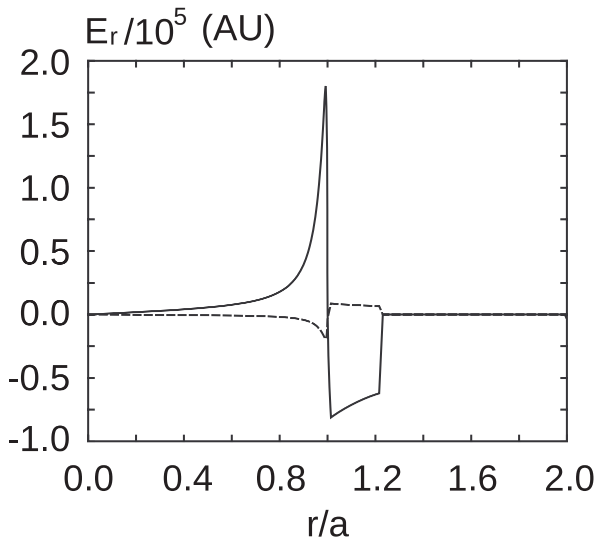
Numerical Results for RWM

- **Comparison between the dispersion relation and the numerical results by TASK/WA**
  - Plasma resistivity η_{plasma} is non-zero in TASK/WA
  - Ideally conducting wall at $r = c$ in TASK/WA

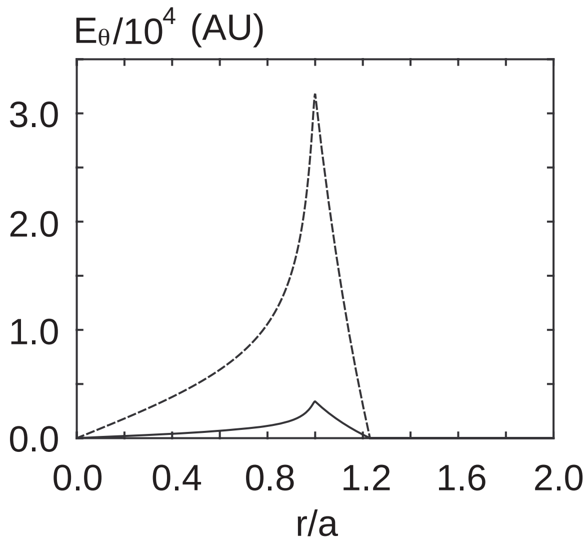
Typical Electric Field Profile (Re/Im)

for $b/a = 1.23$, $c/a = 2.0$, $m = 2$, $n = 1$

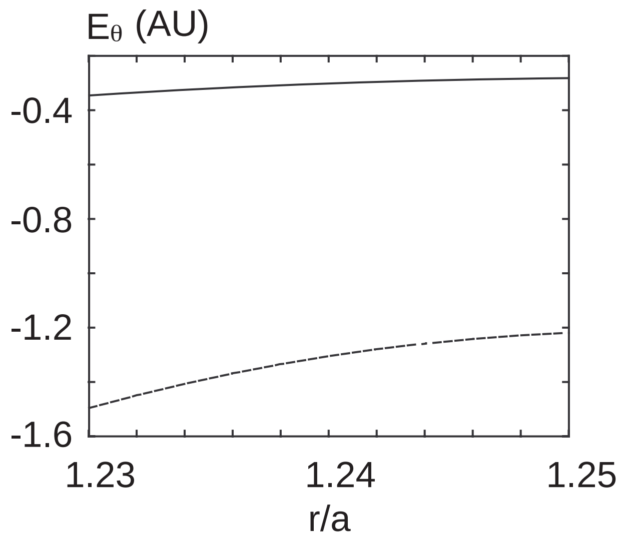
E_r



E_θ



E_θ **in the wall**

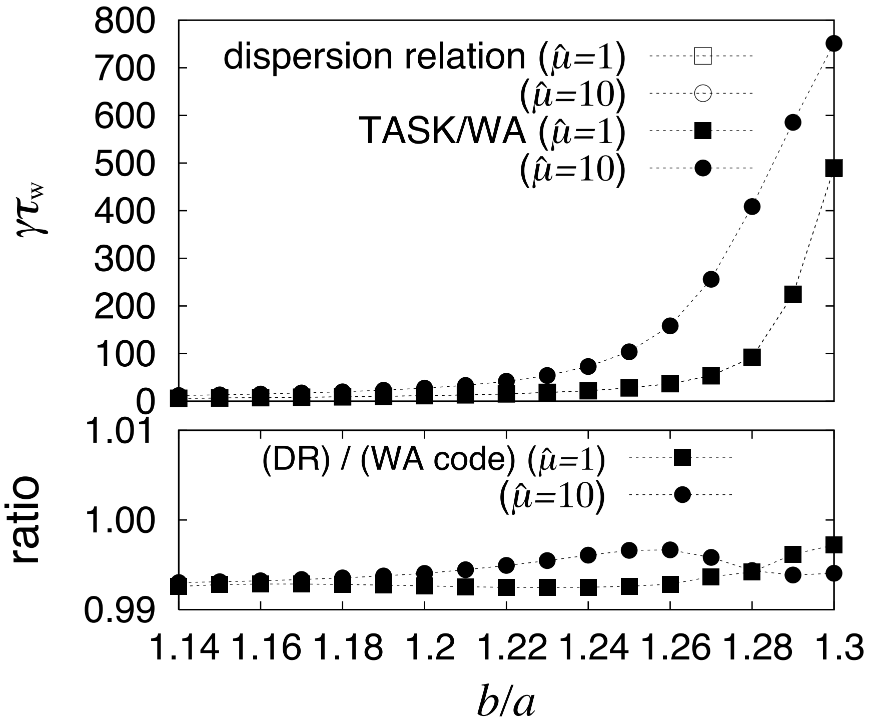


Dependence on the Wall Radius b/a

q profile: Flat

$$q(r) = q_a = 1.35$$

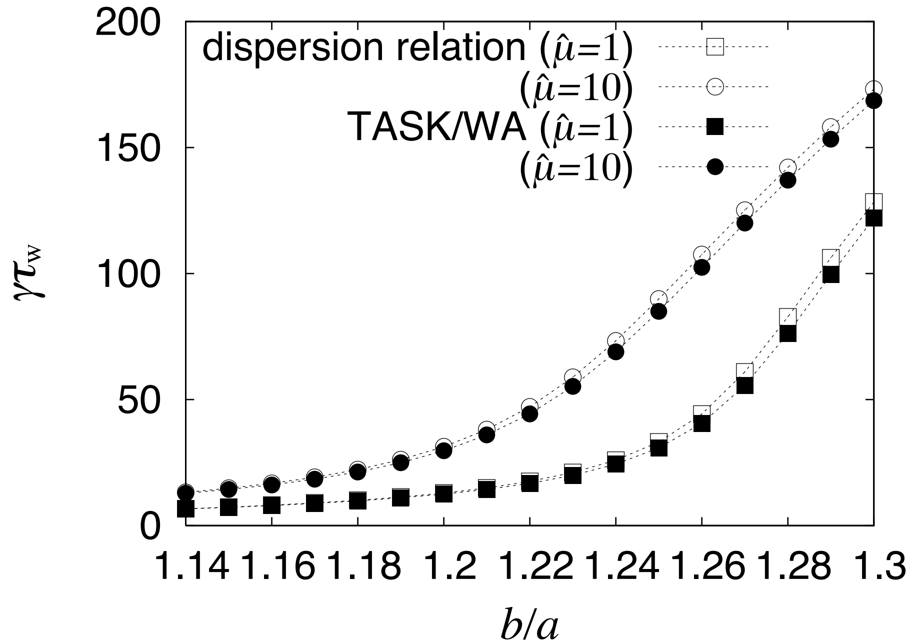
for various values of d/a



q profile: Wesson type

$$q_0 = 0.8,\ q_a = 1.8$$

$$q(r) = q_a \frac{r^2/a^2}{1-(1-r^2/a^2)^{q_a/q_0}}$$



There is a soft threshold of the wall radius b/a.

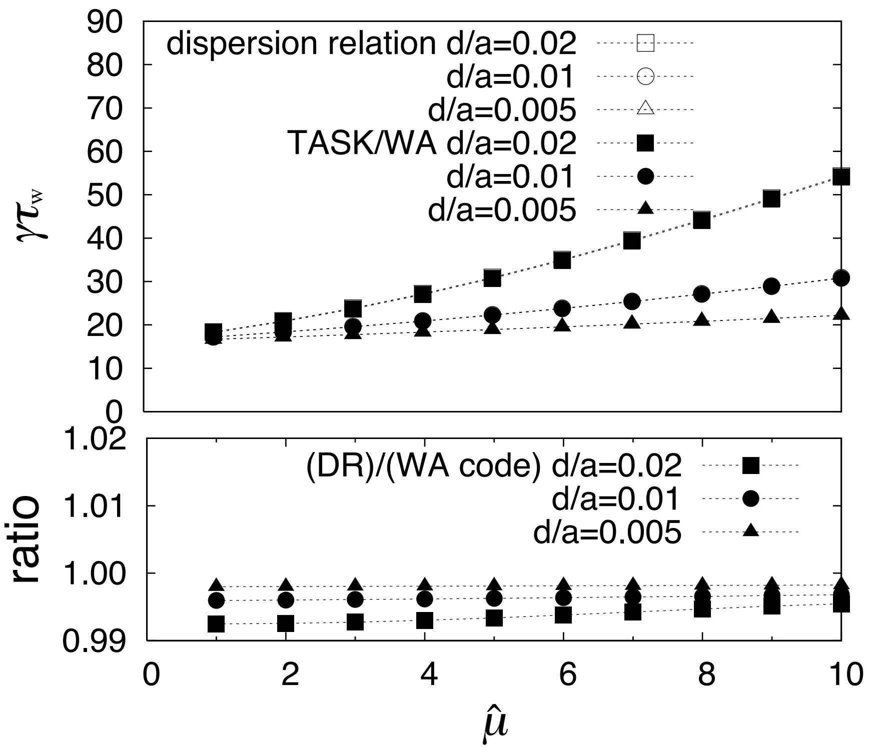
The dispersion relation correctly predicts γ.

Destabilization by $\mu_{\mathrm{Wall}} > 1$ $(b/a = 1.23)$

q profile: Flat

$q(r) = q_a = 1.35$

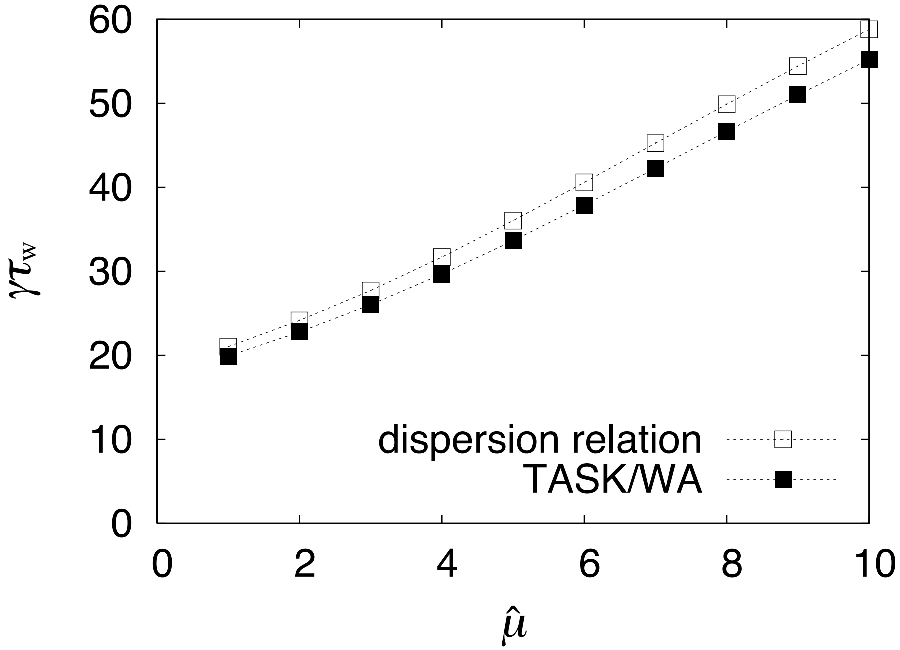
for various values of d/a



q profile: Wesson type

$q_0 = 0.8$, $q_a = 1.8$

wall thickness $d/a = 0.02$



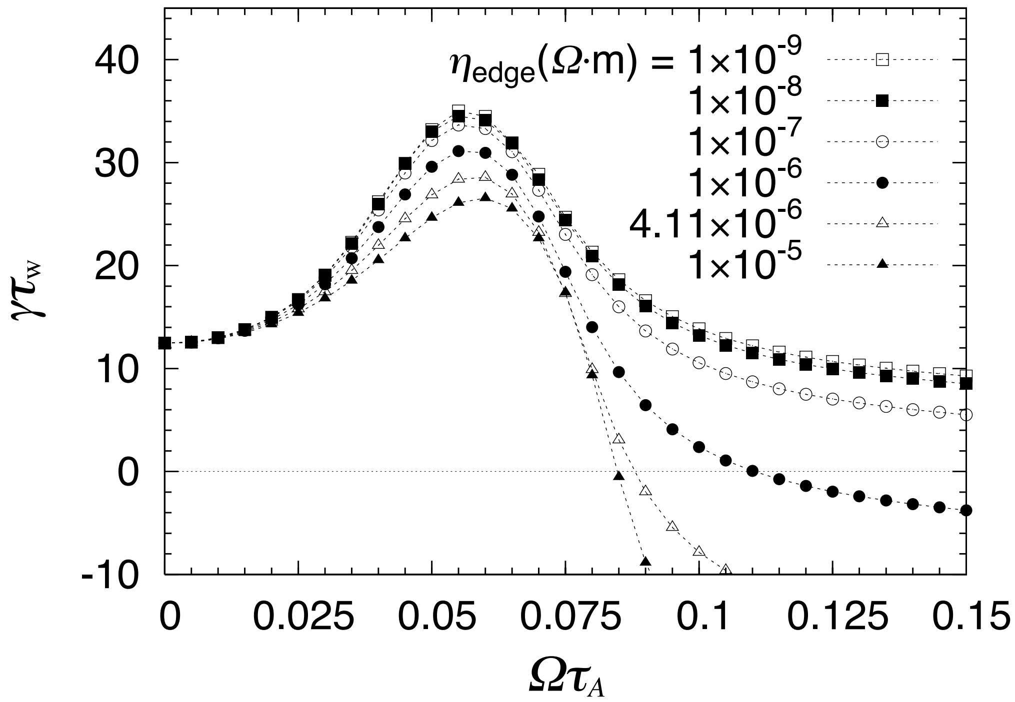
γ increases with the increase of μ_{Wall}.

For flat q profiles, the dispersion relation correctly predicts γ.

Stabilization by Toroidal Rotation

- **Plasma rotation stabilizes RWM** when plasma dissipation exists
- It was confirmed that **the kinetic dissipation is not important** when the plasma is incompressible near the plasma edge.
- q profile: Wesson type ($q_0 = 0.8$C$q_a = 1.8$)

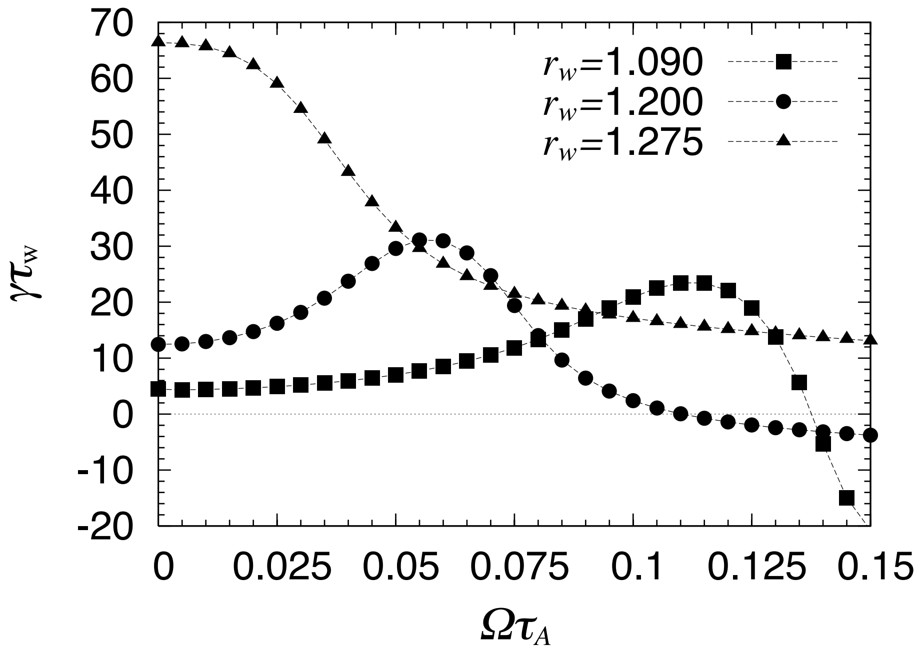
Toroidal rotation dependence of the RWM growth rate for various values of η_{edge}



Comparison of Dissipation Mechanisms

Resistivity only

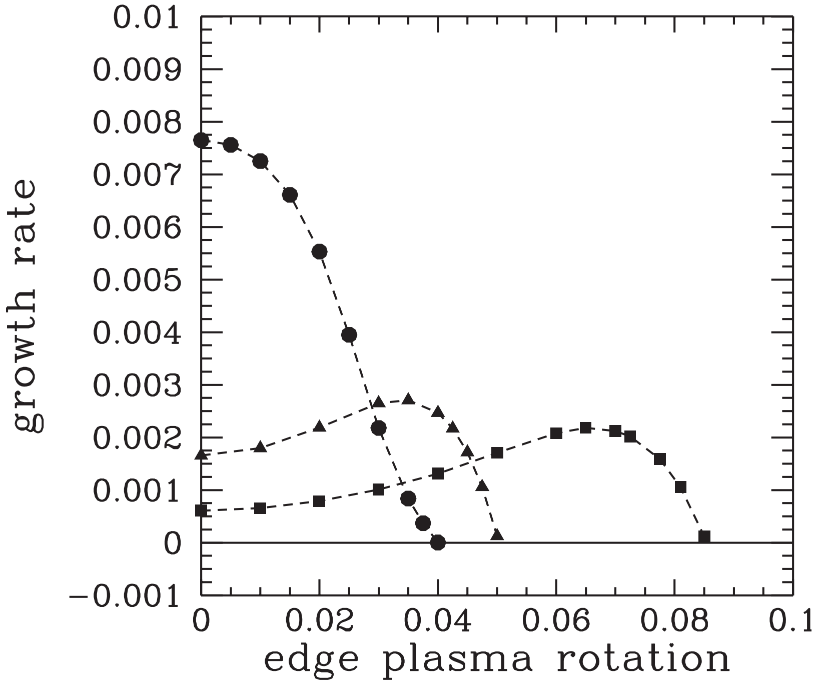
TASK/WA



Resistivity and perp. viscosity

R. Fitzpatrick and A. Aydemir

Nucl. Fusion **36 11 (1996)**



Perpendicular viscosity reduces the growth rate for faster rotation and longer wall distance.

Summary

- We have developed **a full wave code TASK/WA** in order to study the applicability of the full wave analysis to the linear stability study of global modes in a tokamak.
- **Resistive MHD dielectric tensor** was used in the present analysis and the results are compared with previous MHD studies. **Good agreement was shown for cylindrical plasmas and for incompressible modes in tokamak plasmas.**
- We analyzed **the ferromagnetic effect on the resistive wall mode** in a cylindrical plasma using a resistive MHD dielectric tensor.
- The growth rate of RWMs calculated from the dispersion relation agrees well with the results of the TASK/WA code.
- **The critical plasma rotation which stabilizes RWMs increases almost linearly with the increase of $\hat{\mu}$.**